Duality for admissible locally analytic representations

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0. Introduction

In this paper we continue our development of the theory of locally analytic representations of a p-adic locally analytic group G. In our earlier work [ST1,2] we constructed a certain abelian subcategory of the category of modules over the locally analytic distribution algebra $D(G, K)$, where K is a complete discretely valued extension of $\mathbb{Q}_p$. This subcategory of *coadmissible* modules is contravariantly equivalent to the category of admissible locally analytic representations by means of the functor sending a representation V to its strong dual V'_b. The smooth admissible representations correspond to those coadmissible $D(G, K)$-modules which are annihilated by the Lie algebra $\mathfrak{g}$ of G. Here, we study the problem of constructing a contragredient functor on the category of admissible locally analytic representations. In fact, a naive contragredient does not exist. As a best approximation, we construct an involutive "duality" functor from the bounded derived category of $D(G, K)$-modules with coadmissible cohomology to itself. On the subcategory corresponding to complexes of smooth representations, this functor induces the usual smooth contragredient (with a degree shift). Although we construct our functor in general we obtain its involutivity, for technical reasons, only in the case of locally $\mathbb{Q}_p$-analytic groups.

The duality functor we construct in this paper is an extension to coadmissible modules over $D(G, K)$ of the global duality $M \to R\mathrm{Hom}_A(M, A)$ for an Auslander regular ring A. Although $D(G, K)$ itself is not Auslander regular, it "almost" is. More precisely it is a free module over the distribution algebra $D(H, K)$ where H is a compact open subgroup of G. In turn, $D(H, K)$ is, in the terminology of [ST2], a Fréchet-Stein algebra, meaning that it is a projective limit of a family of noetherian Banach algebras $D_r(H, K)$ with flat transition maps. In Section 8 of [ST2] we showed that, if the base field is $\mathbb{Q}_p$, these Banach algebras are Auslander regular rings with global dimension bounded by the dimension of H. Further, we showed that, for a coadmissible module M, the modules $\mathrm{Ext}^*_{D(H,K)}(M, D(H, K))$ are coadmissible. The general theory of coadmissible modules makes it possible to pass back and forth between $D(H, K)$-modules and modules over the Auslander regular Banach algebras $D_r(H, K)$. In addition, a kind of "Shapiro's Lemma" makes it possible to pass back and forth between $D(G, K)$ and $D(H, K)$.

The abelian category of coadmissible modules is filtered by a generalized notion of grade or codimension. Although the duality functor on all coadmissible modules is expressed in terms of a derived category, on the abelian subquotient categories corresponding to the grade filtration, the duality functor is computed as a particular Ext-group.

We now briefly outline the paper. We begin by discussing the smooth contragredient in the setting of coadmissible modules. Such representations correspond to modules over the ring $D^\infty(G, K)$ of locally constant distributions, which is the quotient of $D(G, K)$ by the ideal generated by the Lie algebra $\mathfrak{g}$. We then study the dualizing modules $\mathcal{D}_K^\infty(G)$ and $\mathcal{D}_K(G)$ for smooth and general locally analytic representations respectively. These turn out to be the duals of the compactly supported smooth or locally analytic functions on G. Up to a twist which we suppress here for simplicity, the duality functors on smooth and locally analytic representations are given by $R\mathrm{Hom}_A(., \mathcal{D})$ where A is $D^\infty(G, K)$ or $D(G, K)$ and $\mathcal{D}$ is $\mathcal{D}_K^\infty(G)$ or $\mathcal{D}_K(G)$, though in the smooth case, this is unnecessarily complicated since in fact $\mathcal{D}_K^\infty(G)$ is an injective module. In Section 3 we compare the duality functors in the two categories. We use the (trivial) deRham cohomology of the Lie group G to compute (again, suppressing some twists) $R\mathrm{Hom}_{D(G,K)}(D^\infty(G, K), \mathcal{D}_K(G))$ and from this calculation we obtain the result that the duality on the locally analytic representations induces the usual smooth duality, with a degree shift, on the smooth representations. In Section 4, we establish the involutive nature of the duality functor on the bounded derived category of $D(G, K)$-modules with coadmissible cohomology. Section 5 recalls how this duality respects the subquotient categories of coadmissible modules of fixed codimension. In the final section of the paper we compute the duality functor for principal series representations. We show that a principal series representation induced from a parabolic subgroup P has a single nonvanishing Ext-group in degree equal to the dimension of the subgroup P which is isomorphic to another principal series. It is interesting to compare this with the similar result one obtains in the case of Verma modules ([Kem]).

We thank Matthew Emerton for his suggestions regarding the compatibility of the duality discussed in [ST2] and smooth duality. The first author thanks the University of Illinois at Chicago and the Clay Mathematics Institute for financial support. The second author was supported by NSF grant DMS-0245410.

1. The problem

Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ be complete intermediate fields such that $L/\mathbb{Q}_p$ is finite and K is discretely valued. Throughout let G be a locally L-analytic group of dimension d and let $\mathfrak{g}$ denote its Lie algebra.

We let $D(G, K)$, resp. $D^\infty(G, K)$, be the K-algebra of K-valued locally analytic, resp. locally constant, distributions on G. The group G embeds into $D(G, K)$ as well as into $D^\infty(G, K)$ via the Dirac distributions $g \mapsto \delta_g$. The universal enveloping algebra $U_K(\mathfrak{g}) := U(\mathfrak{g}) \otimes_L K$ is naturally a subalgebra of $D(G, K)$. One has $D^\infty(G, K) = D(G, K)/I_G(\mathfrak{g})$ where $I_G(\mathfrak{g})$ denotes the closed two sided ideal of $D(G, K)$ generated by $\mathfrak{g}$.

Remark 1.1: *i. The group G generates a dense subspace of $D(G, K)$;*

ii. the ideal $I_G(\mathfrak{g})$ is generated by $\mathfrak{g}$ as a left ideal in $D(G,K)$;

iii. $D(G,K) \otimes_{U(\mathfrak{g})} L = D^\infty(G,K)$;

iv. if $\mathfrak{g}$ is semisimple then the ideal $I_G(\mathfrak{g})$ is idempotent.

Proof: i. [ST1] Lemma 3.1. ii. The identity $\delta_g \mathfrak{x} \delta_{g^{-1}} = \mathrm{ad}(g)(\mathfrak{x})$ for any $g \in G$ and $\mathfrak{x} \in \mathfrak{g}$ together with i. imply that $I_G(\mathfrak{g})$ is the closure of the left ideal generated by $\mathfrak{g}$. On the other hand the left ideal $D(G,K)\mathfrak{g}$ certainly is finitely generated. If G is compact it therefore is closed by [ST2] Cor. 3.4.iv and Lemma 3.6. The general case is reduced to the compact case by choosing a compact open subgroup $H \subseteq G$ and observing the locally convex direct sum decompositions

$$D(G,K) = \bigoplus_{g \in G/H} \delta_g D(H,K) \quad \text{and} \quad D(G,K)\mathfrak{g} = \bigoplus_{g \in G/H} \delta_g D(H,K)\mathfrak{g} \ .$$

iii. This follows from ii. iv. This also follows from ii. since, by [Dix] Remark 2.8.8, the ideal generated by $\mathfrak{g}$ in $U(\mathfrak{g})$ is idempotent.

We let $\mathrm{Rep}_K(G)$, resp. $\mathrm{Rep}_K^c(G)$ denote the category of locally analytic G-representations on barrelled locally convex Hausdorff K-vector spaces, resp. on K-vector spaces of compact type, with continuous linear G-maps as morphisms (cf. [ST1] §3). We recall:

– On each G-representation in $\mathrm{Rep}_K(G)$ the G-action extends uniquely to a separately continuous action of the algebra $D(G,K)$.

– The category $\mathrm{Rep}_K^c(G)$ is closed with respect to the passage to closed G-invariant subspaces and their corresponding quotients.

Dually we consider the abelian category $\mathcal{M}_G$ of all (unital left) $D(G,K)$-modules (in the algebraic sense), as well as the category $\mathcal{M}_G^{top}$ of all separately continuous $D(G,K)$-modules on nuclear Fréchet spaces with continuous $D(G,K)$-modules maps as morphisms. By functoriality, passing to the continuous dual is a natural functor

$$\begin{array}{ccc} \mathrm{Rep}_K(G) & \longrightarrow & \mathcal{M}_G \\ V & \longmapsto & V' \ . \end{array}$$

It was proved in [ST1] Cor. 3.3 that the functor

$$(*) \qquad \begin{array}{ccc} \mathrm{Rep}_K^c(G) & \xrightarrow{\simeq} & \mathcal{M}_G^{top} \\ V & \longmapsto & V_b' \end{array}$$

of passing to the strong dual even is an anti-equivalence of categories. (We silently use here, as at many other places in the paper, the map $g \mapsto g^{-1}$ on G to identify left and right $D(G,K)$-modules.)

The objects in $\mathrm{Rep}_K^c(G)$ and in $\mathcal{M}_G^{top}$ already as topological vector spaces are of a quite different nature. This means that the usual construction of a contragredient of a group representation does not preserve the category $\mathrm{Rep}_K^c(G)$.

The question we want to address in this paper is whether there nevertheless is a natural involutory functor on $\mathrm{Rep}_K^c(G)$ which deserves to be considered as a replacement for the usual contragredient. In fact we will restrict our attention to the full subcategory $\mathrm{Rep}_K^a(G)$ of admissible G-representations in $\mathrm{Rep}_K^c(G)$ as constructed in [ST2] §6. We recall that in [ST2] first a full abelian subcategory $\mathcal{C}_G$ (which is entirely algebraic in nature) of $\mathcal{M}_G$, the category of coadmissible $D(G,K)$-modules, is constructed. It then is shown (Lemma 6.1) that $\mathcal{C}_G$ embeds naturally and fully into the topological category $\mathcal{M}_G^{top}$. Finally $\mathrm{Rep}_K^a(G)$ is defined to be the preimage of $\mathcal{C}_G$ under the anti-equivalence $(*)$.

For the sake of completeness we also recall from [ST2] Prop. 6.4 that:

– $\mathrm{Rep}_K^a(G)$ is an abelian category; kernel and image of a morphism in $\mathrm{Rep}_K^a(G)$ are the algebraic kernel and image with the subspace topology.

– Any map in $\mathrm{Rep}_K^a(G)$ is strict and has closed image.

– The category $\mathrm{Rep}_K^a(G)$ is closed with respect to the passage to closed G-invariant subspaces.

An obvious condition which any solution of our problem should satisfy comes about as follows. Let $\mathrm{Rep}_K^\infty(G)$, resp. $\mathrm{Rep}_K^{\infty,a}(G)$ denote the category of smooth, resp. of admissible-smooth, G-representations (over K) in the sense of Jacquet-Langlands (cf. [Cas] §2.1). Equipping a K-vector space with its finest locally convex topology induces a fully faithful embedding

$$\mathrm{Rep}_K^\infty(G) \longrightarrow \mathrm{Rep}_K(G) \ .$$

It was proved in [ST2] Thm. 6.6 that this embedding restricts to a fully faithful embedding

$$\mathrm{Rep}_K^{\infty,a}(G) \longrightarrow \mathrm{Rep}_K^a(G)$$

whose image consists precisely of those admissible locally analytic G-representations which are annihilated by the Lie algebra $\mathfrak{g}$. In the smooth theory one has the contragredient functor

$$\begin{array}{ccc} \mathrm{Rep}_K^\infty(G) & \longrightarrow & \mathrm{Rep}_K^\infty(G) \\ V & \longmapsto & \widetilde{V} \end{array}$$

where the so called smooth dual $\widetilde{V}$ consists of all linear forms in the full linear dual V^* which are smooth, i.e., which are fixed by some open subgroup of G. It restricts to an anti-involution on the subcategory $\mathrm{Rep}_K^{\infty,a}(G)$ (cf. [Cas] Prop. 2.1.10). Surely any construction which one might envisage on $\mathrm{Rep}_K^a(G)$ should be compatible with this one.

It is easy to see that the G-action on any smooth G-representation extends naturally to a $D^\infty(G,K)$-module structure. Because of Remark 1.1.ii we may characterize $\mathrm{Rep}_K^{\infty,a}(G)$ as the full subcategory of $\mathrm{Rep}_K^a(G)$ of those representations

on which the $D(G,K)$-module structure factorizes through a $D^\infty(G,K)$-module structure. If $\mathcal{M}_G^\infty$ denotes the abelian category of (unital left) $D^\infty(G,K)$-modules then passing to the (full) linear dual is a natural functor

$$\begin{array}{ccc} \mathrm{Rep}_K^\infty(G) & \longrightarrow & \mathcal{M}_G^\infty \\ V & \longmapsto & V^* \end{array}$$

which makes the diagram

$$\begin{tikzcd} \mathrm{Rep}_K^\infty(G) \arrow[r, "V\mapsto V^*"] \arrow[d, "finest\ l.c.\ top."'] & \mathcal{M}_G^\infty \arrow[d, "\subseteq"] \\ \mathrm{Rep}_K(G) \arrow[r, "V\mapsto V'"] & \mathcal{M}_G \end{tikzcd}$$

commutative. We also may define the notion of a coadmissible $D^\infty(G,K)$-module. This is based upon the following observation.

Remark 1.2: *For any compact open subgroup $H \subseteq G$ we have:*

i. $D^\infty(H,K)$ is a Fréchet-Stein algebra;

ii. $D^\infty(H,K)$ is coadmissible as a (left or right) $D(H,K)$-module;

iii. a $D^\infty(H,K)$-module is coadmissible if and only if it is coadmissible as a $D(H,K)$-module.

Proof: i. Compare the proof of [ST2] Thm. 6.6.i. ii. This follows from [ST2] Lemma 3.6. iii. This then is a special case of [ST2] Lemma 3.8.

We now define a $D^\infty(G,K)$-module to be coadmissible if it is so as a $D^\infty(H,K)$-module for any compact open subgroup $H \subseteq G$, and we let $\mathcal{C}_G^\infty$ denote the full subcategory of coadmissible $D^\infty(G,K)$-modules in $\mathcal{M}_G^\infty$. We have

$$\mathcal{C}_G^\infty = \mathcal{C}_G \cap \mathcal{M}_G^\infty$$

as well as the commutative diagram

$$\begin{tikzcd} \mathrm{Rep}_K^{\infty,a}(G) \arrow[r, "\simeq"] \arrow[d] & \mathcal{C}_G^\infty \arrow[d] \\ \mathrm{Rep}_K^a(G) \arrow[r, "\simeq"] & \mathcal{C}_G\ . \end{tikzcd}$$

Finally we describe the smooth contragredient on the dual module side. We introduce the K-vector space $C_c^\infty(G,K)$ of all K-valued locally constant functions with compact support on G and its linear dual $\mathcal{D}_K^\infty(G) := C_c^\infty(G,K)^*$. The left

and right translation actions of G on $C_c^\infty(G,K)$ are smooth and therefore extend to $D^\infty(G,K)$-module structures on $C_c^\infty(G,K)$ and hence on $\mathcal{D}_K^\infty(G)$. We fix a left invariant Haar measure $\mu_{Haar} \in \mathcal{D}_K^\infty(G)$. This allows us to introduce, for any compact open subgroup $U \subseteq G$, the function

$$\epsilon_U := \mathrm{vol}_{\mu_{Haar}}(U)^{-1} \cdot \text{char. function of } U$$

in $C_c^\infty(G,K)$. One easily checks that

$$\begin{array}{rcl} C_c^\infty(G,K) & \longrightarrow & D^\infty(G,K) \\ \varphi & \longmapsto & \mu_\varphi := \mu_{Haar}(\varphi \cdot .) \end{array}$$

is a left G-equivariant embedding. We also need the locally constant modulus character $\delta_G : G \longrightarrow \mathbb{Q}^\times \subseteq K^\times$ of G given by

$$\delta_G(g) = [gUg^{-1} : gUg^{-1} \cap U]/[U : gUg^{-1} \cap U]$$

where U is a fixed compact open subgroup of G. In the following we will let δ_G also denote the one dimensional K-vector space K viewed as a $D^\infty(G,K)$-bimodule with G acting trivially from the left and via the character δ_G from the right. Finally we need the functor

$$\begin{array}{rcl} \mathcal{M}_G^\infty & \longrightarrow & \mathrm{Rep}_K^\infty(G) \\ M & \longmapsto & M^{sm} := \{m \in M : Um = m \text{ for some open subgroup } U \subseteq G\} \ . \end{array}$$

Lemma 1.3: *For any module M in $\mathcal{M}_G^\infty$ we have the natural G-equivariant isomorphism*

$$\begin{array}{rcl} M^{sm} & \xrightarrow{\cong} & (C_c^\infty(G,K) \otimes_K \delta_G) \otimes_{D^\infty(G,K)} M \\ m & \longmapsto & (\epsilon_U \otimes 1) \otimes m \qquad \text{if } Um = m \ . \end{array}$$

Proof: The inverse map is given by $(\varphi \otimes 1) \otimes m \longmapsto \mu_\varphi m$.

Lemma 1.4: *The diagram*

$$\begin{array}{ccc} \mathrm{Rep}_K^\infty(G) & \xrightarrow{V \mapsto V^*} & \mathcal{M}_G^\infty \\ {\scriptstyle V\mapsto \widetilde{V}} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G)\otimes_K \delta_G^*)} \\ \mathrm{Rep}_K^\infty(G) & \xrightarrow{V \mapsto V^*} & \mathcal{M}_G^\infty \end{array}$$

is commutative.

Proof: (Recall that $\mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G)\otimes_K \delta_G^*)$ is considered as a $D^\infty(G,K)$-module via the right multiplication on the target.) We need to establish a natural isomorphism between $\mathrm{Hom}_K((V^*)^{sm},K)$ and

$$\mathrm{Hom}_{D^\infty(G,K)}(V^*,\mathcal{D}_K^\infty(G)\otimes_K \delta_G^*)$$
$$= \mathrm{Hom}_K((C_c^\infty(G,K)\otimes_K \delta_G)\otimes_{D^\infty(G,K)} V^*,K)\ .$$

For this it certainly is sufficient to find a G-equivariant natural isomorphism

$$(V^*)^{sm} \cong (C_c^\infty(G,K)\otimes_K \delta_G)\otimes_{D^\infty(G,K)} V^*$$

before passing to the linear dual. But this is a special case of Lemma 1.3.

Because the smooth contragredient is an anti-involution on $\mathrm{Rep}_K^{\infty,a}(G)$ it is clear from Lemma 1.4 that the functor $\mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G)\otimes_K \delta_G^*)$ restricts to an anti-involution on $\mathcal{C}_G^\infty$.

2. Dualizing modules

In this section we will introduce an object analogous to $\mathcal{D}_K^\infty(G)$ in the locally analytic context. We define the locally convex K-vector space $C_c^{an}(G,K)$ of K-valued locally analytic functions with compact support on G by picking a compact open subgroup $H \subseteq G$ and by setting

$$C_c^{an}(G,K) := \bigoplus_{g\in G/H} C^{an}(gH,K)$$

with the locally convex direct sum topology on the right hand side. By [Fea] 2.2.4 this definition is independent of the choice of H. In fact, whenever $G = \dot{\bigcup\limits_{i\in I}} U_i$ is a disjoint covering by compact open subsets U_i we have

$$C_c^{an}(G,K) = \bigoplus_{i\in I} C^{an}(U_i,K)\ .$$

The space $C_c^{an}(G,K)$ is barrelled ([NFA] Ex. 3 after Cor. 6.16) and the left and right translation actions of G on it are locally analytic and hence extend to separately continuous $D(G,K)$-module structures. By functoriality we have corresponding $D(G,K)$-module structures on the strong dual

$$\mathcal{D}_K(G) := C_c^{an}(G,K)'_b\ .$$

Remark 2.1: i. In both actions each individual element in $D(G,K)$ acts by a continuous endomorphism on $\mathcal{D}_K(G)$.

ii. If G is second countable then G/H is countable for any compact open subgroup $H \subseteq G$. Hence $C_c^{an}(G,K)$ is of compact type by [ST1] Prop. 1.2(ii). In this situation [ST1] Cor. 3.3 says that both $D(G,K)$-actions on $\mathcal{D}_K(G)$ are separately continuous.

If $H \subseteq G$ is any compact open subgroup then, by [NFA] 9.10, we have

$$\mathcal{D}_K(G) = \prod_{g \in G/H} D(gH, K) \ .$$

The projection map $\ell_{G,H} : \mathcal{D}_K(G) \longrightarrow D(H,K)$ onto the factor $D(H,K)$ in this decomposition is a canonical $D(H,K)$-bimodule homomorphism. Moreover, we have

$$(*) \qquad m = (\delta_g \ell_{G,H}(\delta_{g^{-1}} m))_{g \in G/H} \qquad \text{for any } m \in \mathcal{D}_K(G).$$

Lemma 2.2: *For any (left) $D(G,K)$-module X the map*

$$\begin{array}{ccc} \mathrm{Hom}_{D(G,K)}(X, \mathcal{D}_K(G)) & \xrightarrow{\cong} & \mathrm{Hom}_{D(H,K)}(X, D(H,K)) \\ F & \longmapsto & \ell_{G,H} \circ F \end{array}$$

is bijective and right $D(H,K)$-equivariant.

Proof: By writing X as the cokernel of a map between free $D(G,K)$-modules it suffices to consider the case $X = D(G,K)$. Then the left hand side is equal to $\mathcal{D}_K(G)$ and the map becomes

$$\begin{array}{ccc} \mathcal{D}_K(G) & \longrightarrow & \mathrm{Hom}_{D(H,K)}(D(G,K), D(H,K)) \\ m & \longmapsto & \mathcal{L}_m(\lambda) := \ell_{G,H}(\lambda m) \ . \end{array}$$

The injectivity of this map is immediate from $(*)$. For the surjectivity we note that an element $\mathcal{L}$ in the right hand side is determined by its values $\mathcal{L}(\delta_{g_0})$ for g_0 running over a set of representatives for the cosets in $H \setminus G$. Define $m := (\delta_g \mathcal{L}(\delta_{g^{-1}}))_{g \in G/H}$. Then

$$\mathcal{L}_m(\delta_{g_0}) = \ell_{G,H}(\delta_{g_0} m) = \delta_{g_0} \delta_{g_0^{-1}} \mathcal{L}(\delta_{g_0}) = \mathcal{L}(\delta_{g_0})$$

and hence $\mathcal{L}_m = \mathcal{L}$.

Proposition 2.3: *For any bounded above complex $X^\cdot$ of $D(G,K)$-modules and any compact open subgroup $H \subseteq G$ we have a natural $D(H,K)$-equivariant isomorphism*

$$\mathrm{Ext}^*_{D(G,K)}(X^\cdot, \mathcal{D}_K(G)) \cong \mathrm{Ext}^*_{D(H,K)}(X^\cdot, D(H,K)) \ .$$

Proof: Since $D(G,K)$ is free over $D(H,K)$ both sides can be computed using a projective resolution of $X^\cdot$ as complex of $D(G,K)$-modules. The assertion then is a consequence of the previous lemma.

For the algebra $D^\infty(G,K)$ we have the following much stronger fact.

Proposition 2.4: *The left $D^\infty(G,K)$-module $\mathcal{D}_K^\infty(G)$ is injective.*

Proof: Fixing a compact open subgroup $H \subseteq G$ we obtain, in a way completely analogous to the proof of Prop. 2.3, that

$$\mathrm{Ext}^*_{D^\infty(G,K)}(Y, \mathcal{D}_K^\infty(G)) \cong \mathrm{Ext}^*_{D^\infty(H,K)}(Y, D^\infty(H,K))$$

for any $D^\infty(G,K)$-module Y. This reduces us to proving that $D^\infty(H,K)$ is a self-injective ring. But $D^\infty(H,K)$ is the projective limit

$$D^\infty(H,K) = \varprojlim_N K[H/N]$$

of the algebraic group rings $K[H/N]$ where N runs over the open normal subgroups of H. It easily follows that $D^\infty(H,K)$ is a direct product of finite dimensional simple K-algebras and as such is self-injective by [Lam] Chap. I Cor. (3.11B).

The fact that such a vanishing result is not available over $D(G,K)$ forces us to work on the level of derived categories. As usual we let $D^b(\mathcal{A})$, for any abelian category $\mathcal{A}$, denote its bounded derived category (which here is understood to be the derived category of all complexes in $\mathcal{A}$ with only finitely many nonzero cohomology objects). Moreover, whenever $\mathcal{A}_0 \subseteq \mathcal{A}$ is a full abelian subcategory closed under extensions we have the triangulated subcategory $D^b_{\mathcal{A}_0}(\mathcal{A})$ of $D^b(\mathcal{A})$ consisting of all those complexes whose cohomology objects lie in $\mathcal{A}_0$. For technical reasons we also will need the "bounded below" versions of these categories which, as usual, are denoted by replacing the superscript "b" by "+"

It is a simple consequence of [ST2] Remark 3.2 that the abelian subcategories $\mathcal{C}_G^\infty$ in $\mathcal{M}_G^\infty$ and $\mathcal{C}_G$ in $\mathcal{M}_G$ are closed under extensions. Hence we have the triangulated subcategories $D^b_{\mathcal{C}_G^\infty}(\mathcal{M}_G^\infty)$ in $D^b(\mathcal{M}_G^\infty)$ and $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ in $D^b(\mathcal{M}_G)$ available.

Since $\mathcal{D}_K^\infty(G) \otimes_K \delta_G^*$ is a $D^\infty(G,K)$-bimodule the functor

$$R\mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G) \otimes_K \delta_G^*) : D^b(\mathcal{M}_G^\infty) \longrightarrow D^+(\mathcal{M}_G^\infty)$$

is well defined. Of course, by Prop. 2.4 (note that a twist preserves injectivity), it is simply given by $\mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G) \otimes_K \delta_G^*)$. The functors $.^*$ and $\tilde{.}$ of passing to the full linear and the smooth dual, respectively, are exact on smooth representations and therefore pass directly to derived categories where we denote

them by the same symbols. As a consequence of Lemma 1.4 we then have the commutative diagram

(2.5)
$$\begin{tikzcd} D^b(\mathrm{Rep}_K^\infty(G)) \arrow[r, "\cdot^*"] \arrow[d, "\sim"'] & D^b(\mathcal{M}_G^\infty) \arrow[d, "R\mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G)\otimes_K \delta_G^*)"] \\ D^b(\mathrm{Rep}_K^\infty(G)) \arrow[r, "\cdot^*"] & D^b(\mathcal{M}_G^\infty)\ . \end{tikzcd}$$

Correspondingly, for a certain twist $\mathcal{D}_K(G) \otimes_K \mathfrak{d}_G$ of the $D(G,K)$-bimodule $\mathcal{D}_K(G)$, to be defined in the next section, we have the functor

$$R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G) : D^b(\mathcal{M}_G) \longrightarrow D^+(\mathcal{M}_G)$$

In the next section we will establish the relation between these two RHom-functors.

3. Lie algebra cohomology

We first recall the standard complexes which compute Lie algebra (co)homology for an arbitrary $\mathfrak{g}$-module X (over K) (cf. [CE] Chap. XIII§8). Let $\bigwedge^{\cdot}\mathfrak{g}$ denote, as usual, the exterior algebra over $\mathfrak{g}$. The Lie algebra cohomology $H^*(\mathfrak{g}, X)$ is the cohomology of the (bounded) complex which in degree q is

$$C^q(\mathfrak{g}; X) := \mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, X)$$

and whose differential is given by

$$\begin{aligned} dc(\mathfrak{x}_0,\ldots,\mathfrak{x}_q) := &\sum_{s<t}(-1)^{s+t+1}c([\mathfrak{x}_s,\mathfrak{x}_t],\mathfrak{x}_1,\ldots,\widehat{\mathfrak{x}_s},\ldots,\widehat{\mathfrak{x}_t},\ldots,\mathfrak{x}_q) \\ &+\sum_s(-1)^{s+1}\mathfrak{x}_s c(\mathfrak{x}_1,\ldots,\widehat{\mathfrak{x}_s},\ldots,\mathfrak{x}_q)\ . \end{aligned}$$

Correspondingly the Lie algebra homology $H_*(\mathfrak{g}, X)$ is the homology of the complex which in degree q is

$$C_q(\mathfrak{g}; X) := \bigwedge^q \mathfrak{g} \otimes_L X$$

and whose differential is given by

$$\begin{aligned} \partial(\mathfrak{x}_1\wedge\ldots\wedge\mathfrak{x}_q\otimes x) := &\sum_{s<t}(-1)^{s+t}[\mathfrak{x}_s,\mathfrak{x}_t]\wedge\mathfrak{x}_1\wedge\ldots\wedge\widehat{\mathfrak{x}_s}\wedge\ldots\wedge\widehat{\mathfrak{x}_t}\wedge\ldots\wedge\mathfrak{x}_q\otimes x \\ &+\sum_s(-1)^{s+1}\mathfrak{x}_1\wedge\ldots\wedge\widehat{\mathfrak{x}_s}\wedge\ldots\wedge\mathfrak{x}_q\otimes\mathfrak{x}_s x\ . \end{aligned}$$

The starting point of our investigation is the following basic fact.

Proposition 3.1: *With respect to the natural either left or right* $\mathfrak{g}$*-module structure on* $D(G,K)$ *we have*

$$H_q(\mathfrak{g}, D(G,K)) = \begin{cases} D^\infty(G,K) & \text{if } q = 0, \\ 0 & \text{if } q > 0 \end{cases}$$

and

$$H_q(\mathfrak{g}, \mathcal{D}_K(G)) = \begin{cases} \mathcal{D}^\infty_K(G) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

Proof: By symmetry it suffices to consider the left $\mathfrak{g}$-module structure. Furthermore, for any compact open subgroup $H \subseteq G$ we have the $\mathfrak{g}$-invariant decompositions $D(G,K) = \bigoplus_{g \in H\backslash G} D(H,K)\delta_g$ and $\mathcal{D}_K(G) = \prod_{g\in H\backslash G} D(H,K)\delta_g$. Since Lie algebra homology commutes with arbitrary direct sums and direct products we may in fact assume that G is compact. We consider now the deRham complex

$$0 \longrightarrow C^{an}(G,K) = A^0(G,K) \xrightarrow{d} A^1(G,K) \xrightarrow{d} \ldots \xrightarrow{d} A^d(G,K) \longrightarrow 0$$

of K-valued global locally analytic differential forms on the locally analytic manifold G. By the usual Poincaré lemma it is an exact resolution of the space $C^\infty(G,K)$ of locally constant functions on G. Since the tangent bundle $TG = \mathfrak{g} \times G$ on G is trivial we have $A^q(G,K) = \mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, C^{an}(G,K))$. This identifies (up to a sign) the deRham complex with the cohomological standard complex for the $\mathfrak{g}$-module $C^{an}(G,K)$ (cf. [BW] VII.1.1) and proves that $H^0(\mathfrak{g}, C^{an}(G,K)) = C^\infty(G,K)$ and $H^q(\mathfrak{g}, C^{an}(G,K)) = 0$ for $q > 0$. But because of the reflexivity of the vector space of compact type $C^{an}(G,K)$ we can go one step further and have

$$A^q(G,K) = \mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, C^{an}(G,K)) = \mathrm{Hom}^{cont}_K(\bigwedge^q \mathfrak{g} \otimes_L D(G,K), K)$$

where Hom^{cont}_K on the right hand side refers to the continuous linear forms. Hence the deRham complex in fact is the continuous dual of the homological standard complex for the $\mathfrak{g}$-module $D(G,K)$. This homological standard complex is a complex of finitely generated free right $D(G,K)$-modules. Its differentials therefore are continuous and strict maps ([ST2] paragraph between Lemma 3.6 and Prop. 3.7). Using the Hahn-Banach theorem we now see that the continuous dual of $H_*(\mathfrak{g}, D(G,K))$ is the cohomology $H^*(\mathfrak{g}, C^{an}(G,K))$ of the deRham complex which we computed already.

By Remark 1.1.iii we have

$$D(G,K) \otimes_{U_K(\mathfrak{g})} K = D^\infty(G,K) \ .$$

It follows that the diagram of functors

$$\begin{tikzcd} \mathcal{M}_G \arrow[r, "\mathrm{Hom}_{D(G,K)}(D^\infty(G,K),.)"] \arrow[d, "\mathrm{forget}"'] & \mathcal{M}_G^\infty \arrow[d, "\mathrm{forget}"] \\ \mathrm{Mod}(U_K(\mathfrak{g})) \arrow[r, "\mathrm{Hom}_{U_K(\mathfrak{g})}(K,.)"] & \mathrm{Vec}_K \end{tikzcd}$$

is commutative where $\mathrm{Mod}(U_K(\mathfrak{g}))$ denotes the category of (unital left) $U_K(\mathfrak{g})$-modules and Vec_K the category of K-vector spaces . If we choose a projective resolution $P.$ of K in $\mathrm{Mod}(U_K(\mathfrak{g}))$ then Prop. 3.1 implies that $D(G,K)\otimes_{U_K(\mathfrak{g})} P.$ is a projective resolution of $D^\infty(G,K)$ in $\mathcal{M}_G$. Choosing now also an injective resolution $I^\cdot$ of an X in $\mathcal{M}_G$ we obtain the following sequence of identities

$$\begin{aligned} R\mathrm{Hom}_{U_K(\mathfrak{g})}(K, \mathrm{forget}(X)) &\sim \mathrm{Hom}_{U_K(\mathfrak{g})}(P., \mathrm{forget}(X)) \\ &\sim \mathrm{Hom}_{D(G,K)}(D(G,K) \otimes_{U_K(\mathfrak{g})} P., X) \\ &\sim \mathrm{forget}(\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), I^\cdot)) \\ &\sim \mathrm{forget}(R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), X)) \end{aligned}$$

in the derived category of Vec_K. Hence we have

$$\mathrm{Ext}^q_{D(G,K)}(D^\infty(G,K), X) = \mathrm{Ext}^q_{U_K(\mathfrak{g})}(K, X) = H^q(\mathfrak{g}, X)$$

for any $q \geq 0$ (we suppress the forgetful functors in the notation) and, in particular, $\mathrm{Ext}^q_{D(G,K)}(D^\infty(G,K), X) = H^q(\mathfrak{g}, X) = 0$ for $q > d$. Therefore the corresponding total right derived functors form a commutative diagram on the level of bounded derived categories:

$$\begin{tikzcd} D^b(\mathcal{M}_G) \arrow[r, "R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K),.)"] \arrow[d, "\mathrm{forget}"'] & D^b(\mathcal{M}_G^\infty) \arrow[d, "\mathrm{forget}"] \\ D^b(\mathrm{Mod}(U_K(\mathfrak{g}))) \arrow[r, "R\mathrm{Hom}_{U_K(\mathfrak{g})}(K,.)"] & D^b(\mathrm{Vec}_K) \end{tikzcd}$$

On the other hand, since the functor $\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), .) : \mathcal{M}_G \longrightarrow \mathcal{M}_G^\infty$ preserves injective objects, we obviously have the adjointness relations

$$\mathrm{Hom}_{D^b(\mathcal{M}_G)}(Y^\cdot, X^\cdot) = \mathrm{Hom}_{D^b(\mathcal{M}_G^\infty)}(Y^\cdot, R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), X^\cdot))$$

and

$$(+) \qquad \begin{aligned} &R\mathrm{Hom}_{D(G,K)}(Y^\cdot, X^\cdot) = \\ &\qquad R\mathrm{Hom}_{D^\infty(G,K)}(Y^\cdot, R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), X^\cdot)) \end{aligned}$$

in $D^b(\mathrm{Vec}_K)$ and in particular

$$\mathrm{Ext}^*_{D(G,K)}(Y^\cdot, X^\cdot) = \mathrm{Ext}^*_{D^\infty(G,K)}(Y^\cdot, R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), X^\cdot))$$

for any $Y^\cdot$ in $D^b(\mathcal{M}^\infty_G)$ and any $X^\cdot$ in $D^b(\mathcal{M}_G)$.

Unfortunately it does not seem to be the case that $R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), .)$ "restricts" to a functor from $D^b(\mathcal{C}_G)$ into $D^b(\mathcal{C}^\infty_G)$.

But there is a little bit more which can be said about this functor. A particular finitely generated free resolution of K in $\mathrm{Mod}(U_K(\mathfrak{g}))$ is given by the homological standard complex

$$U_K(\mathfrak{g}) \otimes_L \bigwedge^d \mathfrak{g} \xrightarrow{\partial} \ldots \xrightarrow{\partial} U_K(\mathfrak{g}) \otimes_L \bigwedge^1 \mathfrak{g} \xrightarrow{\partial} U_K(\mathfrak{g}) \otimes_L \bigwedge^0 \mathfrak{g} \longrightarrow K$$

for $U_K(\mathfrak{g})$ as a right $\mathfrak{g}$-module. Base extending this complex to $D(G,K)$, by Prop. 3.1, is exact and therefore provides the finitely generated free "standard" resolution

$$(*) \qquad D(G,K) \otimes_L \bigwedge^d \mathfrak{g} \xrightarrow{\partial} \ldots \xrightarrow{\partial} D(G,K) \otimes_L \bigwedge^0 \mathfrak{g} \longrightarrow D^\infty(G,K)$$

of $D^\infty(G,K)$ in $\mathcal{M}_G$. We recall that the differential ∂ is given by

$$\begin{aligned}\partial(\lambda \otimes \mathfrak{x}_1 \wedge \ldots \wedge \mathfrak{x}_q) = &\sum_{s<t} (-1)^{s+t} \lambda \otimes [\mathfrak{x}_s, \mathfrak{x}_t] \wedge \mathfrak{x}_1 \wedge \ldots \wedge \widehat{\mathfrak{x}_s} \wedge \ldots \wedge \widehat{\mathfrak{x}_t} \wedge \ldots \wedge \mathfrak{x}_q \\ &+ \sum_s (-1)^{s+1} \lambda \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge \ldots \wedge \widehat{\mathfrak{x}_s} \wedge \ldots \wedge \mathfrak{x}_q \ .\end{aligned}$$

We claim that the resolution $(*)$ carries a natural commuting right $D(G,K)$-module structure which on $D^\infty(G,K)$ is the obvious one. The group $G \subseteq D(G,K)$ obviously acts on $D(G,K) \otimes_L \bigwedge^q \mathfrak{g}$ from the right by

$$(\lambda \otimes \mathfrak{x}_1 \wedge \ldots \wedge \mathfrak{x}_q)\delta_g := \lambda\delta_g \otimes \mathrm{ad}(g^{-1})(\mathfrak{x}_1) \wedge \ldots \wedge \mathrm{ad}(g^{-1})(\mathfrak{x}_q) \ .$$

One easily checks that the differentials ∂ respect this G-action.

Lemma 3.2: *The above G-action on $(*)$ extends uniquely to a separately continuous $D(G,K)$-module structure.*

Proof: See the Appendix. The equivariance of the differentials follows by continuity.

The usual argument with double complexes now shows that, for any bounded complex $X^\cdot$ of $D(G,K)$-modules, we indeed have

$$R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), X^\cdot) \sim \mathrm{Hom}^\cdot_{D(G,K)}(D(G,K) \otimes \bigwedge^\cdot \mathfrak{g}, X^\cdot)$$

in $D^b(\mathcal{M}_G)$. This will enable us to reinterprete this functor in a completely different way. We write in the following $\Delta_G := \bigwedge^d \mathfrak{g} \otimes_L K$ considered as a one dimensional locally analytic G-representation. In particular, Δ_G is a one dimensional $D(G,K)$-module given by an algebra homomorphism which, by abuse of notation, we also write $\Delta_G : D(G,K) \longrightarrow K$. In the most interesting case where G is open in the group of K-valued points of a connected reductive K-group this homomorphism Δ_G in fact is trivial. But there is no point in restricting to this case in the following. As will be shown in the Appendix, the tensor product $\Delta_G \otimes_K X$ with any other $D(G,K)$-module X is defined as a $D(G,K)$-module. Since Δ_G is one dimensional a more explicit way to describe the tensor product $\Delta_G \otimes_K X$ is to say that it is the pull back of the $D(G,K)$-module X via the algebra automorphism

$$\begin{array}{rcl} \alpha_{\Delta_G} : D(G,K) & \longrightarrow & D(G,K) \\ \lambda & \longmapsto & \alpha_{\Delta_G}(\lambda)(\psi) := \lambda(\Delta_G \psi) \ . \end{array}$$

By our above standard resolution the $D^\infty(G,K)$-module $\mathrm{Ext}^d_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K X)$ is the cokernel of the map

$$\begin{array}{c} \mathrm{Hom}_{D(G,K)}(D(G,K) \otimes_L \bigwedge^{d-1} \mathfrak{g}, \Delta_G \otimes_K X) \\ \big\downarrow {\scriptstyle \partial^*} \\ \mathrm{Hom}_{D(G,K)}(D(G,K) \otimes_L \bigwedge^{d} \mathfrak{g}, \Delta_G \otimes_K X) \ . \end{array}$$

For the target we have the natural isomorphism

$$\begin{array}{ccl} X & \xrightarrow{\cong} & \mathrm{Hom}_{D(G,K)}(D(G,K) \otimes_L \bigwedge^d \mathfrak{g}, \Delta_G \otimes_K X) \\ x & \longmapsto & [\lambda \otimes \delta \mapsto \lambda(\delta \otimes x)] \ . \end{array}$$

It in fact is an isomorphism of $D(G,K)$-modules where the left $D(G,K)$-module structure on the right hand side is induced by the right $D(G,K)$-module structure on $D(G,K) \otimes_L \bigwedge^d \mathfrak{g}$; this is easily checked by using the above explicit description of the tensor product modules. To compute the cokernel we fix an L-basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ of $\mathfrak{g}$, and observe right away the elementary identity

$$\begin{aligned} &\textstyle\sum_{i<j} (-1)^{i+j} [\mathfrak{x}_i, \mathfrak{x}_j] \wedge \mathfrak{x}_1 \wedge \ldots \wedge \widehat{\mathfrak{x}_i} \wedge \ldots \wedge \widehat{\mathfrak{x}_j} \wedge \ldots \wedge \mathfrak{x}_d \\ &\qquad = \textstyle\sum_s (-1)^s \mathrm{tr}(\mathrm{Ad}(\mathfrak{x}_s)) \mathfrak{x}_1 \wedge \ldots \wedge \widehat{\mathfrak{x}_s} \wedge \ldots \wedge \mathfrak{x}_d \ . \end{aligned}$$

If write an element $f \in \mathrm{Hom}_{D(G,K)}(D(G,K) \otimes_L \bigwedge^{d-1} \mathfrak{g}, \Delta_G \otimes_K X) = \mathrm{Hom}_L(\bigwedge^{d-1} \mathfrak{g}, \Delta_G \otimes_K X)$ as $f(.) = \mathfrak{x}_1 \wedge \ldots \wedge \mathfrak{x}_d \otimes f_X(.)$ then it is straightforward from the above identity that

$$\partial^* f(\mathfrak{x}_1 \wedge \ldots \wedge \mathfrak{x}_d) = \mathfrak{x}_1 \wedge \ldots \wedge \mathfrak{x}_d \otimes \sum_s (-1)^{s+1} \mathfrak{x}_s f_X(\mathfrak{x}_1 \wedge \ldots \wedge \widehat{\mathfrak{x}_s} \wedge \ldots \wedge \mathfrak{x}_d) \ .$$

In other words, under the above identification of the target with X the image $\partial^* f$ becomes the element $\sum_s (-1)^{s+1} \mathfrak{x}_s f_X(\mathfrak{x}_1 \wedge \ldots \wedge \widehat{\mathfrak{x}_s} \wedge \ldots \wedge \mathfrak{x}_d)$ in X. Since the values of f_X can be chosen arbitrarily this means that the cokernel identifies naturally with $X/\mathfrak{g}X = X/D(G,K)\mathfrak{g}X = D^\infty(G,K) \otimes_{D(G,K)} X$. Hence we have established a natural isomorphism of $D^\infty(G,K)$-modules

$$(1) \qquad \mathrm{Ext}^d_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K X) \cong D^\infty(G,K) \otimes_{D(G,K)} X \ .$$

We repeat that

$$(2) \qquad \mathrm{Ext}^q_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K X) = 0 \qquad \text{for } q > d \ .$$

Let V be any K-vector space and consider $D(G,K) \otimes_K V$ as a left $D(G,K)$-module in the obvious way. Since

$$\begin{aligned} &\mathrm{forget}(\mathrm{Ext}^*_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K D(G,K) \otimes_K V)) \\ &\quad = H^*(\mathfrak{g}, \Delta_G \otimes_K D(G,K) \otimes_K V) \\ &\quad = H_{d-*}(\mathfrak{g}, D(G,K) \otimes_K V) = H_{d-*}(\mathfrak{g}, D(G,K)) \otimes_K V \\ &\quad = 0 \end{aligned}$$

for $* \neq d$, where the second, resp. third, identity comes from [CE] XIII Ex. 15, resp. from Prop. 3.1, we have

$$(3) \qquad \mathrm{Ext}^q_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K D(G,K) \otimes_K V) = 0 \qquad \text{for } q \neq d \ .$$

Finally we observe that any $D(G,K)$-module has a resolution by free $D(G,K)$-modules of the form $D(G,K) \otimes_K V$ (use inductively the natural surjection $D(G,K) \otimes_K X \to X$). In this situation, with (1) – (3), general homological algebra ([Har] I.7.4) tells us that $\mathrm{Ext}^*_{D(G,K)}(D^\infty(G,K), .)$ is the left derived functor of $\mathrm{Ext}^d_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K .) \cong D^\infty(G,K) \otimes_{D(G,K)}$. This establishes the following fact.

Proposition 3.3: *There is a natural isomorphism of $D^\infty(G,K)$-modules*

$$\mathrm{Ext}^*_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K X) \cong \mathrm{Tor}^{D(G,K)}_{d-*}(D^\infty(G,K), X)$$

for any $D(G,K)$-module X.

For our purpose of understanding duality the following special case of these considerations is the most interesting. We introduce the one dimensional tensor product

$$\mathfrak{d}_G := \Delta_G \otimes_K \delta^*_G$$

viewed as a $D(G,K)$-bimodule with $D(G,K)$ acting trivially from the right and through the product character $\Delta_G \cdot \delta_G$ from the left.

Remark 3.4: By [B-GAL] Chap. III§3.16 Cor. to Prop. 55 we have $\delta_G = |\Delta_G|_L$ where $|.|_L$ denotes the normalized absolute value of the field L.

Proposition 3.5: *$R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)$ is naturally quasi-isomorphic to $\mathcal{D}_K^\infty(G) \otimes_K \delta_G^*$ concentrated in degree d.*

Proof: Since $\mathfrak{g}$ acts trivially on δ_G^* a computation analogous to the one between formulae (2) and (3) and based upon the second part of Prop. 3.1 shows that

$$\mathrm{Ext}^q_{D(G,K)}(D^\infty(G,K), \Delta_G \otimes_K \mathcal{D}_K(G) \otimes_K \delta_G^*) = 0 \qquad \text{for } q \neq d \ ,$$

i.e., that the complex in question is concentrated in degree d. Its cohomology in degree d, by formula (1), is naturally isomorphic to $D^\infty(G,K) \otimes_{D(G,K)} (\mathcal{D}_K(G) \otimes_K \delta_G^*)$. Using Remark 1.1 we obtain

$$\begin{aligned} D^\infty(G,K) &\otimes_{D(G,K)} (\mathcal{D}_K(G) \otimes_K \delta_G^*) \\ &= (\mathcal{D}_K(G) \otimes_K \delta_G^*)/\mathfrak{g}(\mathcal{D}_K(G) \otimes_K \delta_G^*) \\ &= (\mathcal{D}_K(G) \otimes_K \delta_G^*)/(\mathfrak{g}\mathcal{D}_K(G) \otimes_K \delta_G^*) \\ &= (\mathcal{D}_K(G)/\mathfrak{g}\mathcal{D}_K(G)) \otimes_K \delta_G^* \\ &= \mathcal{D}_K^\infty(G) \otimes_K \delta_G^* \end{aligned}$$

with the last identity coming from Prop. 3.1.

Corollary 3.6: *The diagram*

$$\begin{tikzcd} D^b(\mathcal{M}_G^\infty) \arrow[r, "\mathrm{can}"] \arrow[d, "R\mathrm{Hom}_{D^\infty(G,K)}(.,\mathcal{D}_K^\infty(G)\otimes_K \delta_G^*[-d])"'] & D^b(\mathcal{M}_G) \arrow[d, "R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G)"] \\ D^b(\mathcal{M}_G^\infty) \arrow[r, "\mathrm{can}"] & D^+(\mathcal{M}_G) \end{tikzcd}$$

is commutative.

Proof: This follows from Prop. 3.5 and the special case

$$\begin{aligned} &R\mathrm{Hom}_{D(G,K)}(Y^\cdot, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) = \\ &\quad R\mathrm{Hom}_{D^\infty(G,K)}(Y^\cdot, R\mathrm{Hom}_{D(G,K)}(D^\infty(G,K), \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)) \end{aligned}$$

of our earlier adjointness relation (+) provided we show that the latter holds true actually in $D^+(\mathcal{M}_G)$. For this we consider the $D(G,K) - D^\infty(G,K)$-bimodule $D^\infty(G,K)$ as a module over the ring $D(G,K) \otimes_K D^\infty(G,K)$ and fix a resolution $F. \xrightarrow{\sim} D^\infty(G,K)$ by free $D(G,K) \otimes_K D^\infty(G,K)$-modules. The point is that this remains a free resolution if only considered as (left) $D(G,K)$-modules or (right) $D^\infty(G,K)$-modules, respectively. If in addition we choose a projective resolution $P^\cdot \xrightarrow{\sim} Y^\cdot$ in $\mathcal{M}_G^\infty$ then the right hand side of the above identity is computed by the complex of $D(G,K)$-modules

$$\mathrm{Hom}^\cdot_{D^\infty(G,K)}(P^\cdot, \mathrm{Hom}_{D(G,K)}(F., \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)) \ .$$

But this complex is equal to the complex

$$\mathrm{Hom}_{D(G,K)}(F. \otimes_{D^\infty(G,K)} P^\cdot, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) \ .$$

Since $F. \otimes_{D^\infty(G,K)} P^\cdot \xrightarrow{\sim} D^\infty(G,K) \otimes_{D^\infty(G,K)} Y^\cdot = Y^\cdot$ is a free resolution of $Y^\cdot$ in $\mathcal{M}_G$ the second Hom-complex computes the left hand side of our identity.

Corollary 3.7: *The diagram*

$$\begin{array}{ccc} D^b(\mathrm{Rep}_K^\infty(G)) & \xrightarrow{\ \cdot^*\ } & D^b(\mathcal{M}_G) \\ \tilde{\cdot}[-d] \downarrow & & \downarrow R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G) \\ D^b(\mathrm{Rep}_K^\infty(G)) & \xrightarrow{\ \cdot^*\ } & D^+(\mathcal{M}_G) \ . \end{array}$$

is commutative.

Proof: Combine (2.5) and Cor. 3.6.

Appendix: The "comultiplication" on $D(G,K)$

In fact, there is no true comultiplication on $D(G,K)$ as will become clear presently. In particular, for two (left) $D(G,K)$-modules X_1 and X_2 the tensor product $X_1 \otimes_K X_2$ does not carry in general a natural $D(G,K)$-module structure. But we do have the following. Let $L \subseteq K$ be two extension fields of $\mathbb{Q}_p$ such that $L/\mathbb{Q}_p$ is finite and K is spherically complete, and let M and N be two paracompact locally L-analytic manifolds. Furthermore let, as usual, the symbols $(\underset{K,\iota}{\widehat{\otimes}})\ \underset{K,\iota}{\otimes}$ and $(\underset{K,\pi}{\widehat{\otimes}})\ \underset{K,\pi}{\otimes}$ stand for the (completed) inductive and projective, respectively, tensor product of two locally convex K-vector spaces (cf. [NFA] §17).

Lemma A.1: *If N is compact then we have*

$$C^{an}(M \times N, K) = C^{an}(M, C^{an}(N,K)) \ .$$

Proof: This is an exercise, which we leave to the reader, in the construction of the locally convex vector spaces $C^{an}(.,.)$ based upon the observation (cf. [Fea] 2.1.2) of a corresponding fact for Banach space valued power series.

Proposition A.2: *Suppose that M is compact, and that V is a K-vector space of compact type; we then have*

$$C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V = C^{an}(M,V) \ .$$

Proof: Apply [Fea] 2.4.3 to the bilinear map $C^{an}(M,K) \times V \to C^{an}(M,V)$ to see that the map is continuous and has dense image. If we show that $C^{an}(M,V)$ induces the projective tensor topology on $C^{an}(M,K) \otimes_K V$, then by completeness we have a topological isomorphism. By the integration theorem ([ST1] Thm. 2.2), we have a linear isomorphism $C^{an}(M,V) \to \mathcal{L}(D(M,K),V)$. At the same time, using the fact that V is the strong dual of a Fréchet space, we see by the discussion before Prop. 20.13 of [NFA] that we have

$$C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V \xrightarrow{\cong} D(M,K)' \mathop{\widehat{\otimes}}_{K,\pi} V \xrightarrow{\cong} \mathcal{L}_b(D(M,K),V) \ .$$

Therefore the map $C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V \to C^{an}(M,V)$ is bijective, as well as continuous. To complete the proof, we wish to apply the open mapping theorem and for this ([NFA] 8.8) it suffices to show that $C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V$ is a direct limit of Banach spaces; we in fact show that it is of compact type. Indeed, $D(M,K)$ and V_b' being reflexive Fréchet spaces, are nuclear ([NFA] 20.7) and therefore ([NFA] 19.11 and 20.4, 20.13, 20.14) $D(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V_b'$ is again nuclear and reflexive, and

$$(D(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V_b')_b' = D(M,K)_b' \mathop{\widehat{\otimes}}_{K,\pi} V = C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V \ .$$

As the strong dual of a nuclear Frechet space, $C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} V$ is of compact type by [ST1] Thm. 1.3. (For a different proof compare [Eme] Prop. 2.1.28.)

Proposition A.3: $D(M \times N, K) = D(M,K) \mathop{\widehat{\otimes}}_{K,\iota} D(N,K)$.

Proof: We first assume that M and N are compact. Recall that the Fréchet space $D(M,K)$ is the strong dual of the reflexive space of compact type $C^{an}(M,K)$ (cf. [ST1] §2). Combining A.1 and A.2 we have

$$C^{an}(M \times N, K) = C^{an}(M,K) \mathop{\widehat{\otimes}}_{K,\pi} C^{an}(N,K) \ .$$

Using reflexivity in addition we therefore obtain

$$\begin{aligned} D(M\times N,K) &= C^{an}(M\times N,K)'_b \\ &= [C^{an}(M,K)\mathop{\widehat{\otimes}}_{K,\pi} C^{an}(N,K)]'_b \\ &= [D(M,K)'_b \mathop{\widehat{\otimes}}_{K,\pi} D(N,K)'_b]'_b \ . \end{aligned}$$

By [NFA] 20.13 and 20.14 the last term is equal to

$$([D(M,K)\mathop{\widehat{\otimes}}_{K,\pi} D(N,K)]'_b)'_b = D(M,K)\mathop{\widehat{\otimes}}_{K,\pi} D(N,K) \ .$$

Since for Fréchet spaces inductive and projective tensor product topology coincide ([NFA] 17.6) our assertion

$$D(M\times N,K) = D(M,K)\mathop{\widehat{\otimes}}_{K,\iota} D(N,K)$$

follows in the compact case.

In the general case the assumption on paracompactness implies ([Sch] Kap. II Satz 8.6) that there are decompositions $M = \dot{\bigcup}_{i\in I} M_i$ and $N = \dot{\bigcup}_{j\in J} N_j$ into disjoint unions of open compact submanifolds M_i and N_j, respectively. Since the completed inductive tensor product commutes with arbitrary locally convex direct sums by an obvious nonarchimedean version of [Gro] Chap. I§3.1 Prop. 14.I we may deduce the general case now as follows:

$$\begin{aligned} D(M\times N,K) &= \bigoplus_{i\in I}\bigoplus_{j\in J} D(M_i\times N_j,K) \\ &= \bigoplus_{i,j}[D(M_i,K)\mathop{\widehat{\otimes}}_{K,\iota} D(N_j,K)] \\ &= [\bigoplus_i D(M_i,K)]\mathop{\widehat{\otimes}}_{K,\iota}[\bigoplus_j D(N_j,K)] \\ &= D(M,K)\mathop{\widehat{\otimes}}_{K,\iota} D(N,K) \ . \end{aligned}$$

As a consequence we see that for $G := M = N$ the diagonal map $G \longrightarrow G\times G$ induces a continuous map

$$D(G,K) \xrightarrow{\mathrm{diag}_*} D(G\times G,K) = D(G,K)\mathop{\widehat{\otimes}}_{K,\iota} D(G,K) \ .$$

It has all the usual properties of a comultiplication; this can be checked on Dirac distributions (where it is obvious) since they generate dense subspaces as recalled in Remark 1.1.i.

Let now X and Y be two (left) $D(G,K)$-modules. Then $X \otimes_K Y$ is a (left) $D(G,K) \otimes_K D(G,K)$-module in the obvious way. We suppose that:

– $\dim_K Y < \infty$, and

– the $D(G,K)$-action on Y is continuous.

Then the annihilator ideal $\operatorname{ann}(Y) \subseteq D(G,K)$ of Y is closed and of finite K-codimension. Moreover $X \otimes_K Y$ in fact is a $D(G,K) \otimes_K D(G,K)/\operatorname{ann}(Y)$-module. Via the map

$$\begin{array}{ccccc} D(G,K) & \xrightarrow{\operatorname{diag}_*} & D(G,K) \underset{K,\iota}{\widehat{\otimes}} D(G,K) & \xrightarrow{\operatorname{pr}} & D(G,K) \underset{K,\iota}{\widehat{\otimes}} D(G,K)/\operatorname{ann}(Y) \\ & & & & \| \\ & & & & D(G,K) \otimes_K D(G,K)/\operatorname{ann}(Y) \end{array}$$

the tensor product $X \otimes_K Y$ acquires a natural "diagonal" (left) $D(G,K)$-module structure.

If X carries a locally convex complete Hausdorff topology with respect to which the $D(G,K)$-action is separately continuous then the diagonal action of $D(G,K)$ on $X \underset{K,\iota}{\widehat{\otimes}} Y = X \underset{K,\pi}{\widehat{\otimes}} Y = X \otimes_K Y$ is the unique separately continuous action which extends the obvious diagonal action of the group G.

Remark A.4: In [ST1] we had to refer to the unpublished Diploma thesis of Féaux de Lacroix for the fact that the convolution product on $D(G,K)$ is well defined and separately continuous. But Prop. A.3 above implies in fact a more precise result. The convolution product is induced by the multiplication map $G \times G \longrightarrow G$ via

$$D(G,K) \times D(G,K) \to D(G,K) \underset{K,\iota}{\widehat{\otimes}} D(G,K) = D(G \times G, K) \xrightarrow{\operatorname{mult}_*} D(G,K) \ .$$

4. Auslander duality - derived categories

As a consequence of Cor. 3.7 we have the commutative diagram:

$$\begin{tikzcd} D^b(\mathrm{Rep}_K^{\infty,a}(G)) \arrow[r, "\cdot^*"] \arrow[d, "\tilde{\cdot}\,[-d]"'] & D^b_{\mathcal{C}_G}(\mathcal{M}_G) \arrow[d, "R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G)"] \\ D^b(\mathrm{Rep}_K^{\infty,a}(G)) \arrow[r, "\cdot^*"] & D^+(\mathcal{M}_G) \ . \end{tikzcd}$$

In this section we do always **assume** that $L = \mathbb{Q}_p$. We will see that then we can replace, as a relatively straightforward consequence of the results in [ST2]§8, the lower right corner in the above diagram by $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$.

For technical reasons we need to fix a compact open subgroup $H \subseteq G$, and we let $D_r(H,K)$, for $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$, be a projective system of noetherian K-Banach algebras which exhibits $D(H,K)$ as a Fréchet-Stein algebra (cf. [ST2]§5). In the following we will make constant use of the fact, without pointing it out each time again, that the ring extensions $D(H,K) \longrightarrow D_r(H,K)$ are flat ([ST2] Remark 3.2). This means that the base extension functor $D_r(H,K) \otimes_{D(H,K)} .$ is exact and hence passes directly to the derived categories. As a piece of additional notation we need the bounded derived category $D^b_{fg}(D_r(H,K))$ of all complexes of (left) $D_r(H,K)$-modules whose cohomology modules are all finitely generated and vanish in all but finitely many degrees.

Proposition 4.1: *For any complex $X^{\cdot}$ in $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ the $D(G,K)$-modules* $\mathrm{Ext}^*_{D(G,K)}(X^{\cdot}, \mathcal{D}_K(G))$ *are coadmissible and vanish in all but finitely many degrees;*

ii. for any r the diagram

$$\begin{array}{ccc}
D^b_{\mathcal{C}_G}(\mathcal{M}_G) & \xrightarrow{R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G))} & D^b_{\mathcal{C}_G}(\mathcal{M}_G) \\
\downarrow{\scriptstyle \text{forget}} & & \downarrow{\scriptstyle \text{forget}} \\
D^b_{\mathcal{C}_H}(\mathcal{M}_H) & \xrightarrow{R\mathrm{Hom}_{D(H,K)}(.,D(H,K))} & D^b_{\mathcal{C}_H}(\mathcal{M}_H) \\
\downarrow{\scriptstyle D_r(H,K)\otimes_{D(H,K)}} & & \downarrow{\scriptstyle D_r(H,K)\otimes_{D(H,K)}} \\
D^b_{fg}(D_r(H,K)) & \xrightarrow{R\mathrm{Hom}_{D_r(H,K)}(.,D_r(H,K))} & D^b_{fg}(D_r(H,K))
\end{array}$$

is commutative (up to natural isomorphism).

Proof: i. Suppose first that $X^{\cdot}$ is a single coadmissible $D(G,K)$-module X concentrated in degree zero. By Prop. 2.3 we have a $D(H,K)$-equivariant isomorphism

$$\mathrm{Ext}^*_{D(G,K)}(X, \mathcal{D}_K(G)) \cong \mathrm{Ext}^*_{D(H,K)}(X, D(H,K)) .$$

According to [ST2] Lemma 8.4 and Thm. 8.9 the right hand side is coadmissible as a $D(H,K)$-module and vanishes for $* > d$. In addition it satisfies

$$\begin{aligned} D_r(H,K) \otimes_{D(H,K)} & \mathrm{Ext}^*_{D(H,K)}(X, D(H,K)) \\ & \cong \mathrm{Ext}^*_{D_r(H,K)}(D_r(H,K) \otimes_{D(H,K)} X, D_r(H,K)) . \end{aligned}$$

Keeping in mind the fact that $\mathcal{C}_G$ is an abelian subcategory of $\mathcal{M}_G$ closed under extensions the case of a general $X^{\cdot}$ in $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ follows now by a straightforward use of the hypercohomology spectral sequence

$$E_2^{a,b} = \mathrm{Ext}^a_{D(G,K)}(h^{-b}(X^{\cdot}), \mathcal{D}_K(G)) \Longrightarrow E^{a+b} = \mathrm{Ext}^{a+b}_{D(G,K)}(X^{\cdot}, \mathcal{D}_K(G)) .$$

It also shows that the formula

$$(*) \qquad \begin{aligned} &D_r(H,K) \otimes_{D(H,K)} \mathrm{Ext}^*_{D(H,K)}(X^{\cdot}, D(H,K)) \\ &\qquad\qquad \cong \mathrm{Ext}^*_{D_r(H,K)}(D_r(H,K) \otimes_{D(H,K)} X^{\cdot}, D_r(H,K)) \ . \end{aligned}$$

holds in general.

ii. We first note that the first and second, resp. the third, horizontal arrow is well defined by assertion i., resp. by [ST2] Thm. 8.9. The upper square is commutative up to a natural isomorphism by Lemma 2.2 (and the fact, already observed in the proof of Prop. 2.3, that any projective $D(G,K)$-module is projective over $D(H,K)$ as well). The lower square is commutative by the formula $(*)$ above.

The twist functor $. \otimes_K \mathfrak{d}_G : \mathcal{M}_G \longrightarrow \mathcal{M}_G$ is an auto-equivalence which respects the subcategory $\mathcal{C}_G$ (cf. [Eme] Prop. 6.1.5).

Corollary 4.2: *The diagram*

$$\begin{tikzcd} D^b(\mathrm{Rep}_K^{\infty,a}(G)) \arrow[r, "\cdot^*"] \arrow[d, "\tilde{\cdot}{[-d]}"'] & D^b_{\mathcal{C}_G}(\mathcal{M}_G) \arrow[d, "R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G)"] \\ D^b(\mathrm{Rep}_K^{\infty,a}(G)) \arrow[r, "\cdot^*"] & D^b_{\mathcal{C}_G}(\mathcal{M}_G) \end{tikzcd}$$

is commutative.

Proof: What remains to be shown, because of Cor. 3.7, is that for any $X^{\cdot}$ in $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ the $D(G,K)$-modules $\mathrm{Ext}^*_{D(G,K)}(X^{\cdot}, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G)$ are coadmissible and vanish in all but finitely many degrees. But we may write $X^{\cdot} = Y^{\cdot} \otimes_K \mathfrak{d}_G$ for some other complex $Y^{\cdot}$ in $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ and then have the natural isomorphism

$$\mathrm{Ext}^*_{D(G,K)}(X^{\cdot}, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) \cong \mathrm{Ext}^*_{D(G,K)}(Y^{\cdot}, \mathcal{D}_K(G)) \ .$$

Since the action of $D(G,K)$ on $\mathfrak{d}_G$ from the right is trivial this isomorphism in particular is $D(G,K)$-equivariant. The claim therefore is a consequence of Prop. 4.1.i.

The left perpendicular arrow in the diagram of Cor. 4.4 is an anti-involution. What about the right perpendicular arrow?

Proposition 4.3: *The natural transformation*

$$X^{\cdot} \xrightarrow{\cong} R\mathrm{Hom}_{D(G,K)}(R\mathrm{Hom}_{D(G,K)}(X^{\cdot}, \mathcal{D}_K(G)), \mathcal{D}_K(G))$$

on $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$ is an isomorphism.

Proof: *Step 1:* The natural transformation

$$Y^{\cdot} \xrightarrow{\cong} R\mathrm{Hom}_{D_r(H,K)}(R\mathrm{Hom}_{D_r(H,K)}(Y^{\cdot}, D_r(H,K)), D_r(H,K))$$

on $D^b_{fg}(D_r(H,K))$ is an isomorphism.

Since $D_r(H,K)$ has finite global dimension $Y^{\cdot}$ can be represented by a perfect complex $P^{\cdot}$, i.e., by a complex consisting of finitely generated projective $D_r(H,K)$-modules at most finitely many of which are nonzero (cf. [Ha2] III.12.3). The assertion therefore is equivalent to the natural homomorphism of complexes

$$P^{\cdot} \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{D_r(H,K)}(\mathrm{Hom}^{\cdot}_{D_r(H,K)}(P^{\cdot}, D_r(H,K)), D_r(H,K))$$

being a quasi-isomorphism. But this is a well known fact and can be seen by a straightforward double complex argument.

Step 2: The natural transformation

$$X^{\cdot} \xrightarrow{\cong} R\mathrm{Hom}_{D(H,K)}(R\mathrm{Hom}_{D(H,K)}(X^{\cdot}, D(H,K)), D(H,K))$$

on $D^b_{\mathcal{C}_H}(\mathcal{M}_H)$ is an isomorphism.

We fix a projective resolution $P^{\cdot} \xrightarrow{\sim} X^{\cdot}$ and an injective resolution $D(H,K) \xrightarrow{\sim} I^{\cdot}$ in $\mathcal{M}_H$. We have to show that the natural homomorphism of complexes

$$P^{\cdot} \longrightarrow \mathrm{Hom}^{\cdot}_{D(H,K)}(\mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)), I^{\cdot})$$

is a quasi-isomorphism, i.e., that the maps

$$h^*(P^{\cdot}) \longrightarrow h^*(\mathrm{Hom}^{\cdot}_{D(H,K)}(\mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)), I^{\cdot}))$$

are isomorphisms. Since both sides are coadmissible it suffices to show that the maps

$$\begin{aligned} &D_r(H,K) \otimes_{D(H,K)} h^*(P^{\cdot}) \\ &\qquad \longrightarrow D_r(H,K) \otimes_{D(H,K)} h^*(\mathrm{Hom}^{\cdot}_{D(H,K)}(\mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)), I^{\cdot})) \end{aligned}$$

are isomorphisms. The left hand side obviously is equal to $h^*(D_r(H,K) \otimes_{D(H,K)} P^{\cdot})$. To rewrite the right hand side we fix an injective resolution

$$D_r(H,K) \otimes_{D(H,K)} I^{\cdot} \xrightarrow{\sim} J^{\cdot}$$

in the category of (left) $D_r(H,K)$-modules. Applying formula $(*)$ in the proof of Prop. 4.1 to the complex $\mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K))$ which, in $D^b_{\mathcal{C}_H}(\mathcal{M}_H)$, represents $R\mathrm{Hom}_{D(H,K)}(X^{\cdot}, D(H,K))$ we have the natural isomorphism

$$\begin{aligned} &D_r(H,K) \otimes_{D(H,K)} h^*(\mathrm{Hom}^{\cdot}_{D(H,K)}(\mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)), I^{\cdot})) \\ &\qquad \xrightarrow{\cong} h^*(\mathrm{Hom}^{\cdot}_{D(H,K)}(D_r(H,K) \otimes_{D(H,K)} \mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)), J^{\cdot})) . \end{aligned}$$

using formula $(*)$ once more we furthermore have the quasi-isomorphism

$$\begin{aligned} &D_r(H,K) \otimes_{D(H,K)} \mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{D_r(H,K)}(D_r(H,K) \otimes_{D(H,K)} P^{\cdot}, D_r(H,K)) \end{aligned}$$

which induces (cf. [Bor] I.10.5) the quasi-isomorphism

$$\begin{aligned} &\mathrm{Hom}^{\cdot}_{D_r(H,K)}(\mathrm{Hom}^{\cdot}_{D_r(H,K)}(D_r(H,K) \otimes_{D(H,K)} P^{\cdot}, D_r(H,K)), J^{\cdot}) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}^{\cdot}_{D_r(H,K)}(D_r(H,K) \otimes_{D(H,K)} \mathrm{Hom}^{\cdot}_{D(H,K)}(P^{\cdot}, D(H,K)), J^{\cdot}) \ . \end{aligned}$$

Our above map therefore becomes a map

$$\begin{aligned} &h^*(D_r(H,K) \otimes_{D(H,K)} P^{\cdot}) \\ &\qquad \longrightarrow h^*(\mathrm{Hom}^{\cdot}_{D(H,K)}(\mathrm{Hom}^{\cdot}_{D(H,K)}(D_r(H,K) \otimes_{D(H,K)} P^{\cdot}, D(H,K)), J^{\cdot})) \end{aligned}$$

which is easily checked to be the natural transformation for the complex $D_r(H,K) \otimes_{D(H,K)} P^{\cdot}$ treated in Step 1 and hence is an isomorphism by that step.

Step 3: We fix a projective resolution $P^{\cdot} \xrightarrow{\sim} X^{\cdot}$ as $D(G,K)$-modules (and a fortiori as $D(H,K)$-modules) and injective resolutions $\mathcal{D}_K(G) \xrightarrow{\sim} J^{\cdot}$ as $D(G,K)$-modules and $D(H,K) \xrightarrow{\sim} I^{\cdot}$ as $D(H,K)$-modules. On the one hand we then have, by Lemma 2.2, the isomorphism of complexes

$$\begin{array}{rcl} \mathbf{1} : \mathrm{Hom}_{D(G,K)}(P^{\cdot}, \mathcal{D}_K(G)) & \xrightarrow{\cong} & \mathrm{Hom}_{D(H,K)}(P^{\cdot}, D(H,K)) \\ F & \longmapsto & \ell_{G,H} \circ F \end{array} .$$

On the other hand the map $\ell_{G,H}$ extends to a map

$$\begin{tikzcd} \mathcal{D}_K(G) \arrow[r, "\sim"] \arrow[d, "\ell_{G,H}"'] & J^{\cdot} \arrow[d, "\ell"] \\ D(H,K) \arrow[r, "\sim"] & I^{\cdot} \end{tikzcd}$$

of complexes. The natural transformation of our assertion is, as a map of complexes, given as

$$P^{\cdot} \longrightarrow \prod_i \mathrm{Hom}_{D(G,K)}(\mathrm{Hom}_{D(G,K)}(P^{-i}, \mathcal{D}_K(G)), J^{\cdot+i})$$

where $P^{\cdot}$ maps in the obvious way into the factor $\mathrm{Hom}_{D(G,K)}(\mathrm{Hom}_{D(G,K)}(P^{\cdot}, \mathcal{D}_K(G)), J^0)$ with $i = -\cdot$ of the right hand side. A straightforward computation shows that the diagram

$$\begin{tikzcd} P^{\cdot} \arrow[r] \arrow[d, "="'] & \prod_i \mathrm{Hom}_{D(G,K)}(\mathrm{Hom}_{D(G,K)}(P^{-i}, \mathcal{D}_K(G)), J^{\cdot+i}) \arrow[d, "\prod \ell \circ . \circ \mathbf{1}^{-1}"] \\ P^{\cdot} \arrow[r] & \prod_i \mathrm{Hom}_{D(H,K)}(\mathrm{Hom}_{D(H,K)}(P^{-i}, D(H,K)), I^{\cdot+i}) \end{tikzcd}$$

is commutative. By Step 2 the lower horizontal arrow is a quasi-isomorphism. Our assertion therefore is established once we show that the right perpendicular arrow is a quasi-isomorphism. But the latter arrow is the composite of the maps

$$\prod_i \mathrm{Hom}_{D(G,K)}(\mathrm{Hom}_{D(G,K)}(P^{-i}, \mathcal{D}_K(G)), J^{\cdot+i})$$

$$\xrightarrow{\prod \ell\circ.} \prod_i \mathrm{Hom}_{D(H,K)}(\mathrm{Hom}_{D(G,K)}(P^{-i}, \mathcal{D}_K(G)), I^{\cdot+i})$$

and

$$\prod_i \mathrm{Hom}_{D(H,K)}(\mathrm{Hom}_{D(G,K)}(P^{-i}, \mathcal{D}_K(G)), I^{\cdot+i})$$

$$\xrightarrow{\prod .\circ \mathrm{I}^{-1}} \prod_i \mathrm{Hom}_{D(H,K)}(\mathrm{Hom}_{D(G,K)}(P^{-i}, \mathcal{D}_K(G)), I^{\cdot+i})$$

of which the latter even is an isomorphism of complexes. The former is a quasi-isomorphism by Prop. 2.3 applied to the complex $R\mathrm{Hom}_{D(G,K)}(X^\cdot, \mathcal{D}_K(G))$, resp. to a projective resolution of it (observe [Bor] I.10.5).

Corollary 4.4: *The functor*

$$R\mathrm{Hom}_{D(G,K)}(., \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) : D^b_{\mathcal{C}_G}(\mathcal{M}_G) \longrightarrow D^b_{\mathcal{C}_G}(\mathcal{M}_G)$$

is an anti-involution.

Proof: From Prop. 4.3 we know that the functor $R\mathrm{Hom}_{D(G,K)}(., \mathcal{D}_K(G))$ is an anti-involution on $D^b_{\mathcal{C}_G}(\mathcal{M}_G)$. To deal with the twist by $\mathfrak{d}_G$ we let $\mathfrak{d}_G : G \longrightarrow K^\times$ also denote the locally analytic character which describes the G-action on the left $D(G,K)$-module $\mathfrak{d}_G$. A straightforward explicit computation shows that the map

$$\begin{array}{ccc} \mathrm{Hom}_{D(G,K)}(X, \mathcal{D}_K(G)) \otimes_K \mathfrak{d}_G & \xrightarrow{\cong} & \mathrm{Hom}_{D(G,K)}(X, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) \\ F \otimes u & \longmapsto & [x \mapsto F(x)(\mathfrak{d}_G \cdot .) \otimes u] \end{array}$$

is, for any $D(G,K)$-module X, an isomorphism of (left) $D(G,K)$-modules. Using a projective resolution $P^\cdot \xrightarrow{\sim} X^\cdot$ we now may compute

$$\begin{aligned} & R\mathrm{Hom}_{D(G,K)}(R\mathrm{Hom}_{D(G,K)}(X^\cdot, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G), \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) \\ & \quad \cong R\mathrm{Hom}_{D(G,K)}(\mathrm{Hom}^\cdot_{D(G,K)}(P^\cdot, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G), \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) \\ & \quad \cong R\mathrm{Hom}_{D(G,K)}(\mathrm{Hom}^\cdot_{D(G,K)}(P^\cdot, \mathcal{D}_K(G)) \otimes_K \mathfrak{d}_G, \mathcal{D}_K(G) \otimes_K \mathfrak{d}_G) \\ & \quad \cong R\mathrm{Hom}_{D(G,K)}(\mathrm{Hom}^\cdot_{D(G,K)}(P^\cdot, \mathcal{D}_K(G)), \mathcal{D}_K(G)) \\ & \quad \cong R\mathrm{Hom}_{D(G,K)}(R\mathrm{Hom}_{D(G,K)}(X^\cdot, \mathcal{D}_K(G)), \mathcal{D}_K(G)) \\ & \quad \cong X^\cdot . \end{aligned}$$

We do know now that the outer solid arrow rectangle in the diagram

$$\begin{tikzcd}
D^b(\mathrm{Rep}_K^{\infty,a}(G)) \arrow[r] \arrow[d, "\widetilde{.}[-d]"'] & D^b(\mathrm{Rep}_K^{a}(G)) \arrow[r, ".'"] \arrow[d, dashed] & D^b_{\mathcal{C}_G}(\mathcal{M}_G) \arrow[d, "R\mathrm{Hom}_{D(G,K)}(.,\mathcal{D}_K(G)\otimes_K \mathfrak{d}_G)"] \\
D^b(\mathrm{Rep}_K^{\infty,a}(G)) \arrow[r] & D^b(\mathrm{Rep}_K^{a}(G)) \arrow[r, ".'"] & D^b_{\mathcal{C}_G}(\mathcal{M}_G)
\end{tikzcd}$$

is commutative with the perpendicular arrows being anti-involutions. It remains an open question whether there is a natural perpendicular arrow in the middle which makes each square commutative.

5. Auslander duality - abelian categories

We keep in this section the assumption that $L = \mathbb{Q}_p$. For any coadmissible module X in $\mathcal{C}_G$ we define its *grade* or *codimension* by

$$j_{D(G,K)}(X) := \min\{l \geq 0 : \mathrm{Ext}^l_{D(G,K)}(X, \mathcal{D}_K(G)) \neq 0\} \ .$$

As a consequence of Prop. 2.3 we have

$$j_{D(G,K)}(X) = j_{D(H,K)}(X)$$

for any compact open subgroup $H \subseteq G$. Hence the present notion of codimension coincides with the one introduced in [ST2] p.193. From [ST2] Prop. 8.11 we then know that X carries a natural dimension filtration

$$X = \Delta^0(X) \supseteq \Delta^1(X) \supseteq \ldots \supseteq \Delta^{d+1}(X) = 0$$

by $D(G,K)$-submodules having the following properties:

i. Each $\Delta^l(X)$ is a coadmissible $D(G,K)$-module;

ii. a coadmissible $D(G,K)$-submodule $X' \subseteq X$ has codimension $\geq l$ if and only if $X' \subseteq \Delta^l(X)$;

iii. $j_{D(G,K)}(X) = \sup\{l \geq 0 : \Delta^l(X) = X\}$;

iv. all nonzero coadmissible $D(G,K)$-submodules of $\Delta^l(X)/\Delta^{l+1}(X)$ have codimension l.

We define $\mathcal{C}_G^l$ to be the full subcategory in $\mathcal{C}_G$ of all coadmissible modules of codimension $\geq l$. By the above properties

$$\mathcal{C}_G = \mathcal{C}_G^0 \supseteq \mathcal{C}_G^1 \supseteq \ldots \supseteq \mathcal{C}_G^{d+1} = 0$$

is a filtration of the abelian category $\mathcal{C}_G$ by Serre subcategories. We therefore may form the quotient abelian categories $\mathcal{C}_G^l/\mathcal{C}_G^{l+1}$. We remark that $\mathcal{C}_G^\infty \subseteq \mathcal{C}_G^d$ by [ST2] Cor. 8.13. For the purposes of our study of duality we need the following property which is implicit in the results in [ST2] §8.

Lemma 5.1: *$j_{D(G,K)}(\mathrm{Ext}^l_{D(G,K)}(X, \mathcal{D}_K(G))) \geq l$.*

Proof: Let $H \subseteq G$ be a fixed compact open subgroup. By Prop. 2.3 it suffices to show that $j_{D(H,K)}(\mathrm{Ext}^l_{D(H,K)}(X, D(H,K))) \geq l$. With the notations of §4 it furthermore suffices, by [ST2] Lemma 8.4, to see that

$$\mathrm{Ext}^i_{D_r(H,K)}(\mathrm{Ext}^l_{D_r(H,K)}(M, D_r(H,K), D_r(H,K))) = 0$$

for $i < l$ and any finitely generated $D_r(H,K)$-module M. But this is a formal consequence of the Auslander regularity of $D_r(H,K)$ ([ST2] Thm. 8.9).

It follows that the composed functor

$$\mathrm{Ext}^l_{D(G,K)}(., \mathcal{D}_K(G)) : \mathcal{C}_G^l \longrightarrow \mathcal{C}_G^l \longrightarrow \mathcal{C}_G^l/\mathcal{C}_G^{l+1}$$

is well defined and exact. Since it is zero on the Serre subcategory $\mathcal{C}_G^{l+1}$ it induces, by the universal property of quotient categories, an exact functor $\mathcal{C}_G^l/\mathcal{C}_G^{l+1} \longrightarrow \mathcal{C}_G^l/\mathcal{C}_G^{l+1}$.

Proposition 5.2: *The functor*

$$\mathrm{Ext}^l_{D(G,K)}(., \mathcal{D}_K(G)) : \mathcal{C}_G^l/\mathcal{C}_G^{l+1} \longrightarrow \mathcal{C}_G^l/\mathcal{C}_G^{l+1}$$

is an anti-involution.

Proof: By Prop. 4.3 we have a natural isomorphism

$$X \cong h^0(R\mathrm{Hom}_{D(G,K)}(R\mathrm{Hom}_{D(G,K)}(X^\cdot, \mathcal{D}_K(G)), \mathcal{D}_K(G))) \ .$$

The hypercohomology spectral sequence for the right hand side therefore is of the form

$$E_2^{a,b} = \mathrm{Ext}^a_{D(G,K)}(\mathrm{Ext}^{-b}_{D(G,K)}(X, \mathcal{D}_K(G)), \mathcal{D}_K(G)) \Longrightarrow E^{a+b}$$

with $E^{a+b} = X$ for $a+b = 0$ and $E^{a+b} = 0$ otherwise. Let $F^\cdot X$ denote the filtration induced by this spectral sequence on its abutment X, i.e., $F^a X/F^{a+1}X = E_\infty^{a,-a}$. It follows from Lemma 5.1 that all terms of this spectral sequence lie in $\mathcal{C}_G^l$. Moreover, we have $E_2^{a,b} = 0$ for $a < -b$. This implies

$$E_2^{a,-a} \supseteq E_3^{a,-a} \supseteq \ldots \supseteq E_{d+2}^{a,-a} = E_\infty^{a,-a} = F^a X/F^{a+1}X$$

for any $a \geq 0$ and with all these terms vanishing for $a < l$; in particular, $F^l X = X$. It therefore remains to show that the composed map

$$X \longrightarrow X/F^{l+1}X \longrightarrow E_2^{l,-l} = \mathrm{Ext}^l_{D(G,K)}(\mathrm{Ext}^{-l}_{D(G,K)}(X, \mathcal{D}_K(G)), \mathcal{D}_K(G))$$

is an isomorphism in the quotient category $\mathcal{C}_G^l/\mathcal{C}_G^{l+1}$. We have

$$E_{i+1}^{a,-a} = \ker(E_i^{a,-a} \longrightarrow E_i^{a+i,-a-i+1}) \ .$$

But $E_i^{a+i,-a-i+1}$, hence $E_i^{a,-a}/E_{i+1}^{a,-a}$ for $i \geq 2$, and a fortiori $E_2^{a,-a}/E_\infty^{a,-a}$ lie in $\mathcal{C}_G^{a+1}$, by Lemma 5.1. It follows that $F^{l+1}X$ and $E_2^{l,-l}/E_\infty^{l,-l}$ both lie in $\mathcal{C}_G^{l+1}$. (We remark that a slight refinement of this argument shows that the filtration $F^{\cdot}X$ actually coincides with the dimension filtration $\Delta^{\cdot}(X)$.)

6. The locally analytic principal series

In this last section we want to illustrate the preceding theory by computing explicitly the duality functors for a series of genuinely locally analytic representations. From now on G is the group of $\mathbb{Q}_p$-rational points of a connected reductive group over $\mathbb{Q}_p$ (in particular $L = \mathbb{Q}_p$). We fix a parabolic subgroup $P \subseteq G$ as well as a locally analytic character $\chi : P \longrightarrow K^\times$. We will use the same letter for the corresponding algebra homomorphism $\chi : D(P,K) \longrightarrow K$. The P-representation, resp. the $D(P,K)$-module, given by χ will be denoted by K_χ. The locally analytic principal series representation of G corresponding to χ is given as

$$\begin{array}{ll}\mathrm{Ind}_P^G(\chi) := & \text{vector space of all locally analytic functions } f : G \longrightarrow K \\ & \text{such that } f(gp) = \chi^{-1}(p) f(g) \text{ for any } g \in G \text{ and } p \in P\end{array}$$

with G acting by left translation. By [Fea] 4.1.5 the orbit maps $g \mapsto gf$, for $f \in \mathrm{Ind}_P^G(\chi)$, are locally analytic. In the following it will be technically important to fix a maximal compact subgroup $G_0 \subseteq G$ with the property that $G = G_0 P$. We set $P_0 := G_0 \cap P$. Then ([Fea] 4.1.4) restriction of functions induces a G_0-equivariant topological isomorphism

$$\mathrm{Ind}_P^G(\chi) \xrightarrow{\cong} \mathrm{Ind}_{P_0}^{G_0}(\chi)$$

(with the right hand side having the obvious meaning). The space $\mathrm{Ind}_{P_0}^{G_0}(\chi)$ is closed in $C^{an}(G_0, K)$ and hence is of compact type with its continuous dual being a Hausdorff quotient of $D(G_0, K)$ ([ST1] Prop. 1.2(i)). Using [ST2] Lemma 3.6 we conclude that $\mathrm{Ind}_{P_0}^{G_0}(\chi)$ and $\mathrm{Ind}_P^G(\chi)$ are admissible G_0- and G-representations, respectively. We now compute the dual modules. The group

P_0 is topologically finitely generated; we fix a finite system $\{p_i\}_{i\in I}$ of such topological generators. The exact sequence of admissible G_0-representations

$$\begin{array}{rcl} 0 \longrightarrow \mathrm{Ind}_{P_0}^{G_0}(\chi) \xrightarrow{\subseteq} C^{an}(G_0,K) & \longrightarrow & \oplus_I \, C^{an}(G_0,K) \\ f & \longmapsto & (f(.p_i) - \chi^{-1}(p_i)f)_i \end{array}$$

dualizes into the exact sequence of $D(G_0,K)$-modules

$$\begin{array}{rcl} \oplus_I \, D(G_0,K) & \longrightarrow & D(G_0,K) \longrightarrow \mathrm{Ind}_{P_0}^{G_0}(\chi)'_b \longrightarrow 0 \\ (\lambda_i)_i & \longmapsto & \sum_i \lambda_i \delta_{p_i} - \chi^{-1}(p_i)\lambda_i \ . \end{array}$$

Hence

$$\mathrm{Ind}_{P_0}^{G_0}(\chi)'_b = D(G_0,K)/J_{P_0}$$

where J_{P_0} denotes the left ideal of $D(G_0,K)$ generated by $\{\delta_{p_i} - \chi^{-1}(p_i)\delta_1\}_{i\in I}$. On the other hand a completely similar reasoning shows that the very same elements generate the kernel of the algebra homomorphism $\chi^{-1} : D(P_0,K) \longrightarrow K$ as a left ideal. It follows that

$$\mathrm{Ind}_{P_0}^{G_0}(\chi)' = D(G_0,K) \otimes_{D(P_0,K)} K_{\chi^{-1}} \ .$$

Lemma 6.1: *i. The natural map*

$$D(G_0,K) \otimes_{D(P_0,K)} D(P,K) \xrightarrow{\cong} D(G,K)$$

is an isomorphism of $D(G_0,K)$-$D(P,K)$-bimodules;

ii. for any (left) $D(P,K)$-module X the natural map

$$D(G_0,K) \otimes_{D(P_0,K)} X \xrightarrow{\cong} D(G,K) \otimes_{D(P,K)} X$$

is an isomorphism of $D(G_0,K)$-modules;

iii. there is a natural isomorphism of $D(G_0,K)$-modules

$$\mathrm{Tor}_*^{D(P_0,K)}(D(G_0,K),X) \cong \mathrm{Tor}_*^{D(P,K)}(D(G,K),X)$$

for any (left) $D(P,K)$-module X;

iv. in the commutative diagram

$$\begin{array}{ccc} D(G_0,K) \otimes_{D(P_0,K)} K_{\chi^{-1}} & \xrightarrow{\cong} & \mathrm{Ind}_{P_0}^{G_0}(\chi)' \\ \downarrow \cong & & \downarrow \cong \\ D(G,K) \otimes_{D(P,K)} K_{\chi^{-1}} & \xrightarrow{\cong} & \mathrm{Ind}_{P}^{G}(\chi)' \end{array}$$

all four maps are isomorphisms.

Proof: i. This is clear from the decompositions $D(G,K) = \oplus_{p\in P_0\backslash P} D(G_0,K)\delta_p$ and $D(P,K) = \oplus_{p\in P_0\backslash P} D(P_0,K)\delta_p$. ii. This is immediate from the first assertion. iii. Since $D(P,K)$ is free as a $D(P_0,K)$-module a projective resolution of X as a $D(P,K)$-module can be used to compute both sides of the asserted isomorphism. Hence the present assertion is a consequence of the previous one. iv. (Recall that the horizontal arrows are induced by dualizing the inclusions $\mathrm{Ind}_{P_0}^{G_0}(\chi) \subseteq C^{an}(G_0,K)$ and $\mathrm{Ind}_P^G(\chi) \subseteq C^{an}(G,K)$, respectively, the right perpendicular arrow is the dual of the restriction map, and the left perpendicular arrow is induced by the inclusion $D(G_0,K) \subseteq D(G,K)$.) We know already that the upper horizontal and the right perpendicular arrows are isomorphisms. Hence it remains to see, for example, that the left perpendicular arrow is bijective which is a special case of ii.

Our goal is to compute the $D(G,K)$-modules

$$\mathrm{Ext}^*_{D(G,K)}(\mathrm{Ind}_P^G(\chi)', \mathcal{D}_K(G)) \cong \mathrm{Ext}^*_{D(G,K)}(D(G,K) \otimes_{D(P,K)} K_{\chi^{-1}}, \mathcal{D}_K(G)) \ .$$

Part of this computation can be done in greater generality for any (left) $D(P,K)$-module X which satisfies the following condition:

(FIN) As a $D(P_0,K)$-module X has a resolution $P. \xrightarrow{\sim} X$ by finitely generated projective $D(P_0,K)$-modules.

Any such X is coadmissible of course. We begin with a small technical digression into the results of [ST2] to consider compatibilities between $D(G_0,K)$ and its subalgebra $D(P_0,K)$.

Let G_1 be an open normal uniform subgroup of G_0. Given an ordered basis $h_1,\ldots,h_d$ for G_1 we have an explicit family of norms $||\cdot||_r$ ($1/p < r < 1$ and $r \in p^{\mathbb{Q}}$) on the algebra $D(G_1,K)$ defining its Fréchet-Stein structure. The algebra $D(G_0,K)$ is a free, finite rank, left (or right) $D(G_1,K)$-module with basis given by the Dirac distributions for coset representatives of G_0/G_1, and in [ST2] Thm. 5.1 we show that the norms $||\cdot||_r$ extend to $D(G_0,K)$ simply by writing elements of this algebra in a basis and taking the maximum of the norms of the coefficients. The Banach algebras $D_r(G_0,K)$, for $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$, are the completions of $D(G_0,K)$ with respect to the norms $||\cdot||_r$.

Naturally, all of the constructions described here may also be carried out for the group P_0. The following result shows that they may be done "simultaneously" for P_0 and G_0.

Proposition 6.2: *One can choose the family of norms $||\cdot||_r$ on $D(G_0,K)$ so that they define the Fréchet-Stein structure on both $D(G_0,K)$ and its subalgebra $D(P_0,K)$, and so that, for each $1/p < r < 1$ in $p^{\mathbb{Q}}$, the completion $D_r(G_0,K)$ is flat as a left as well as a right $D_r(P_0,K)$-module.*

Proof: The result holds more generally for an arbitrary compact p-adic Lie group H and a closed subgroup $Q \subseteq H$, and we will prove it in this situation. By [DDMS] Ex. 4.14, we may find an open normal uniform subgroup H' of H such that $H' \cap Q = Q'$ is open normal uniform in Q. Furthermore, we may arrange that $(H')^p \cap Q' = Q'^p$. In this situation we find an ordered basis of topological generators $a_1, \ldots, a_d$ for H' such that the first $m := \dim(Q)$ members are an ordered basis for Q' (cf. [DDMS] Prop. 1.9(iii) and Lemma 3.4). Therefore inside the explicit realization of $D(H', K)$ as a noncommutative power series ring in the variables $b_i := a_i - 1$ for $i = 1, \ldots, d$, as in [ST2] §4, the subalgebra $D(Q', K)$ consists of those power series involving only the first m variables. The norms $|| \cdot ||_r$ for $D(H', K)$ restrict to the corresponding ones for $D(Q', K)$. The statement and proof of Thm. 4.5 of [ST2] shows that, on the level of graded rings, the map from $D_r(Q', K)$ to $D_r(H', K)$ is just the inclusion of a polynomial ring in $m + 1$ variables into one of $d + 1$ variables, which is clearly flat. The conditions of Prop. 1.2 of [ST2] being satisfied, this tells us that $D_r(Q', K) \to D_r(H', K)$ is also flat, both on the left and right.

Now following the proof of [ST2] Thm. 5.1, we extend the norms on $D(H', K)$ to $D(H, K)$ by choosing coset representatives for H' in H, writing

$$D(H, K) = \oplus_{h \in H/H'} D(H', K)\delta_h$$

as a free (left) $D(H', K)$ module, and, for each r, taking the maximum of the $|| \cdot ||_r$–norm of the components in this representation. If we further require that representatives in the image of the inclusion map $Q/Q' \hookrightarrow H/H'$ be chosen from Q, then the restriction of $|| \cdot ||_r$ to $D(Q, K) = \oplus_{h \in Q/Q'} D(Q', K)\delta_h$ respects this direct sum. By the flatness result in the uniform case, we know that $D_r(Q, K) \otimes_{D_r(Q', K)} D_r(H', K)$ is flat as a left $D_r(Q, K)$-module. However, a computation with coset representatives shows that $D_r(H, K)$ is a finite direct sum of copies of $D_r(Q, K) \otimes_{D_r(Q', K)} D_r(H', K)$, and is therefore flat as a left $D_r(Q, K)$-module. The right flatness follows in the same way.

Lemma 6.3: *Suppose that the (left) $D(P, K)$-module X satisfies (FIN); we then have:*

i. $\mathrm{Tor}_*^{D(P,K)}(D(G, K), X) = \mathrm{Tor}_*^{D(P_0,K)}(D(G_0, K), X) = 0$ *for* $* > 0$*;*

ii. there is a natural isomorphism of (right) $D(G, K)$-modules

$$\mathrm{Ext}^*_{D(G,K)}(D(G, K) \otimes_{D(P,K)} X, \mathcal{D}_K(G)) \cong \mathrm{Ext}^*_{D(P,K)}(X, \mathcal{D}_K(G)) \ .$$

Proof: i. Because of Lemma 6.1.iii we only need to show the vanishing of the second Tor-groups. Using the projective resolution $P. \xrightarrow{\sim} X$ from (FIN) we have

$$\mathrm{Tor}_*^{D(P_0,K)}(D(G_0, K), X) = h_*(D(G_0, K) \otimes_{D(P_0,K)} P.) \ .$$

But $D(G_0,K)\otimes_{D(P_0,K)} P.$ is a complex of coadmissible $D(G_0,K)$-modules. So its homology is coadmissible which means that its vanishing can be tested on the corresponding coherent sheaves (cf. [ST2] Cor. 3.1). By [ST2] Remark 3.2 we have

$$D_r(G_0,K)\otimes_{D(G_0,K)} h_*(D(G_0,K)\otimes_{D(P_0,K)} P.) = h_*(D_r(G_0,K)\otimes_{D(P_0,K)} P.)\ .$$

Since both ring homomorphisms $D(P_0,K)\longrightarrow D_r(P_0,K)\longrightarrow D_r(G_0,K)$ are flat, the first one by [ST2] Remark 3.2 and the second one by Prop. 6.2, the groups $h_*(D_r(G_0,K)\otimes_{D(P_0,K)} P.)$ vanish in degrees $*>0$.

ii. The assertion is obvious in degree $*=0$. The general case follows from this if we use a projective resolution $Q.\xrightarrow{\sim} X$ of $D(P,K)$-modules to compute the right hand side because $D(G,K)\otimes_{D(P,K)} Q.\xrightarrow{\sim} D(G,K)\otimes_{D(P,K)} X$ then, by i., is a projective resolution which we may use to compute the left hand side.

Proposition 6.4: *Suppose that the (left) $D(P,K)$-module X as well as all the (right) $D(P,K)$-modules $\mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(P))$ satisfy (FIN); then there is a natural isomorphism of (right) $D(G,K)$-modules*

$$\begin{aligned}\mathrm{Ext}^*_{D(G,K)}(D(G,K)\otimes_{D(P,K)} X,\mathcal{D}_K(G)) \cong &\\ &\mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(P))\otimes_{D(P,K)} D(G,K)\ .\end{aligned}$$

Proof: Since $\mathcal{D}_K(P)\subseteq\mathcal{D}_K(G)$ we have, by functoriality, the homomorphism of right $D(G,K)$-modules

$$\begin{array}{rcl}\mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(P))\otimes_{D(P,K)} D(G,K) & \longrightarrow & \mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(G))\\ e\otimes\lambda & \longmapsto & \mathrm{Ext}^*_{D(P,K)}(X,.\cdot\lambda)(e)\ .\end{array}$$

We claim that this map in fact is an isomorphism, which by Lemma 6.3.ii suffices for our assertion. For this we consider the diagram:

$$\begin{tikzcd}
\mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(P))\otimes_{D(P,K)} D(G,K) \arrow[r] & \mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(G)) \arrow[dd, "\cong"', "\mathrm{Ext}^*(X,\ell_{G,G_0})"] \\
\mathrm{Ext}^*_{D(P,K)}(X,\mathcal{D}_K(P))\otimes_{D(P_0,K)} D(G_0,K) \arrow[u, "\cong"'] \arrow[d, "\mathrm{Ext}^*(X,\ell_{P,P_0})\otimes id", "\cong"'] & \\
\mathrm{Ext}^*_{D(P_0,K)}(X,D(P_0,K))\otimes_{D(P_0,K)} D(G_0,K) \arrow[r] & \mathrm{Ext}^*_{D(P_0,K)}(X,D(G_0,K))
\end{tikzcd}$$

The upper left perpendicular arrow is an isomorphism by the right module version of Lemma 6.1.ii. The lower left and the right perpendicular arrows are isomorphisms by Prop. 2.3, resp. an argument entirely analogous to the proof

of Prop. 2.3. The commutativity of this diagram reduces to the commutativity of

$$\begin{tikzcd} \mathcal{D}_K(P) \arrow[r, "\cdot\mu"] \arrow[d, "\ell_{P,P_0}"'] & \mathcal{D}_K(G) \arrow[d, "\ell_{G,G_0}"] \\ D(P_0,K) \arrow[r, "\cdot\mu"] & D(G_0,K) \end{tikzcd}$$

for any $\mu \in D(G_0,K)$ which is obvious. Our claim therefore reduces to the lower horizontal arrow in the first diagram being an isomorphism. Let $P. \xrightarrow{\sim} X$ be the projective resolution from (FIN). We have to show that

$$\begin{array}{c} h^*(\mathrm{Hom}_{D(P_0,K)}(P., D(P_0,K))) \otimes_{D(P_0,K)} D(G_0,K) \\ \Big\downarrow \\ h^*(\mathrm{Hom}_{D(P_0,K)}(P., D(G_0,K))) \end{array}$$

is an isomorphism. Since each term in the complex $P.$ is finitely generated projective we have

$$\mathrm{Hom}_{D(P_0,K)}(P., D(G_0,K)) = \mathrm{Hom}_{D(P_0,K)}(P., D(P_0,K)) \otimes_{D(P_0,K)} D(G_0,K) \ .$$

This finally reduces our claim to the statement that the natural map

$$h^*(Y^{\cdot}) \otimes_{D(P_0,K)} D(G_0,K) \longrightarrow h^*(Y^{\cdot} \otimes_{D(P_0,K)} D(G_0,K))$$

is an isomorphism for the complex $Y^{\cdot} := \mathrm{Hom}_{D(P_0,K)}(P., D(P_0,K))$. The complex $Y^{\cdot}$ consists of finitely generated projective right $D(P_0,K)$-modules. By Prop. 4.1 it is bounded. Moreover, by assumption, its cohomology are (right) $D(P_0,K)$-modules which satisfy (FIN). Therefore this statement is a formal consequence, by a hypercohomology spectral sequence argument, of the right module version of Lemma 6.3.i.

Proposition 6.5: *i.* $\mathrm{Ext}^*_{D(G,K)}(\mathrm{Ind}_P^G(\chi)', \mathcal{D}_K(G)) = 0$ *for* $* \neq \dim(P)$*;*

ii. $\mathrm{Ext}^{\dim(P)}_{D(G,K)}(\mathrm{Ind}_P^G(\chi)', \mathcal{D}_K(G)) \cong \mathrm{Ind}_P^G((\chi\mathfrak{d}_P)^{-1})'$.

Proof: (We emphasize that in ii. we consider both sides, as usual, as left $D(G,K)$-modules.) First of all we have to verify that the $D(P,K)$-module $X := K_{\chi^{-1}}$ satisfies the assumptions of Prop. 6.4. Let $D^c(P_0,K)$ denote the convolution algebra of K-valued continuous distributions (= measures) on the compact group P_0. It is the continuous dual of the Banach space of all K-valued continuous functions on P_0. Alternatively it can be constructed as follows. Let o denote the ring of integers in K and form the completed group ring

$$o[[P_0]] := \varprojlim_N o[P_0/N]$$

where N runs over all open normal subgroups of P_0. Then

$$D^c(P_0, K) \cong K \otimes_o o[[P_0]] \ .$$

The ring $o[[P_0]]$ is known to be noetherian by a straightforward generalization of [Laz] V.2.2.4; hence $D^c(P_0, K)$ is noetherian. The continuous homomorphism $\chi^{-1} : P_0 \longrightarrow o^\times \subseteq K^\times$ extends to an algebra homomorphism $\chi^{-1} : D^c(P_0, K) \longrightarrow K$ which allows us to view $K_{\chi^{-1}}$ as a (left) $D^c(P_0, K)$-module. Since $D^c(P_0, K)$ is noetherian we find a resolution $Q. \xrightarrow{\sim} K_{\chi^{-1}}$ by finitely generated projective $D^c(P_0, K)$-modules. In [ST2] Thm. 5.2 we have shown that the natural ring homomorphism $D^c(P_0, K) \longrightarrow D(P_0, K)$ is flat. For the sake of clarity we point out that the statement of loc. cit. only says that the map $D^c(P_0, K') \longrightarrow D(P_0, K)$ is flat for any subfield $K' \subseteq K$ which is finite over $\mathbb{Q}_p$. But noticing that $o[[P_0]]$ is a linearly compact o-module the same proof actually gives this slightly more general result needed here. (Warning: The ring denoted by $K[[P_0]]$ in loc. cit. in general is dense in but not equal to $D^c(P_0, K)$.) It follows that

$$P. := D(P_0, K) \otimes_{D^c(P_0,K)} Q. \xrightarrow{\sim} D(P_0, K) \otimes_{D^c(P_0,K)} K_{\chi^{-1}}$$

is a resolution by finitely generated projective $D(P_0, K)$-modules. Since, as discussed at the beginning of this section, the kernel of $\chi^{-1} : D(P_0, K) \longrightarrow K$, resp. of $\chi^{-1} : D^c(P_0, K) \longrightarrow K$, is the left ideal generated by the elements $\delta_p - \chi^{-1}(p)\delta_1$, for $p \in P_0$, in the respective ring we have

$$D(P_0, K) \otimes_{D^c(P_0,K)} K_{\chi^{-1}} = K_{\chi^{-1}} \ .$$

Hence $P. \xrightarrow{\sim} K_{\chi^{-1}}$ is a resolution as required in (FIN). Furthermore, by the computation in the proof of Cor. 4.4 we have right $D(P, K)$-module isomorphisms

$$\begin{aligned} \mathrm{Ext}^*_{D(P,K)}(K_{\chi^{-1}}, \mathcal{D}_K(P)) &= \mathrm{Ext}^*_{D(P,K)}(K, \mathcal{D}_K(P) \otimes_K K_\chi) \\ &= \mathrm{Ext}^*_{D(P,K)}(K, \mathcal{D}_K(P) \otimes_K \mathfrak{d}_P) \otimes_K K_{(\chi\mathfrak{d}_P)^{-1}} \ . \end{aligned}$$

Applying Cor. 3.7 to the trivial (and hence smooth) P-representation we obtain

$$\mathrm{Ext}^*_{D(P,K)}(K, \mathcal{D}_K(P) \otimes_K \mathfrak{d}_P) = \begin{cases} K & \text{if } * = \dim(P), \\ 0 & \text{otherwise.} \end{cases}$$

as right $D(P, K)$-modules. We see that

$$\mathrm{Ext}^*_{D(P,K)}(K_{\chi^{-1}}, \mathcal{D}_K(P)) = \begin{cases} K_{(\chi\mathfrak{d}_P)^{-1}} & \text{if } * = \dim(P), \\ 0 & \text{otherwise.} \end{cases}$$

as right $D(P,K)$-modules. In particular these $D(P,K)$-modules satisfy (FIN) as well. We therefore may apply Prop. 6.4 and, together with Lemma 6.1.iv, we obtain

$$\mathrm{Ext}^*_{D(G,K)}(\mathrm{Ind}_P^G(\chi)', \mathcal{D}_K(G)) \cong \mathrm{Ext}^*_{D(P,K)}(K_{\chi^{-1}}, \mathcal{D}_K(P)) \otimes_{D(P,K)} D(G,K)$$

as right $D(G,K)$-modules. Inserting into this the previous computation, converting right into left modules, and using Lemma 6.1.iv once more establishes the assertion.

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